Abstract

In freight transport shippers tend to prefer truck for the shortest trips, rail for medium-distance trips and water for the longest trips. Hence, the price for transport by truck is lowest for short distances and intersects at some point with the price for transport by rail when distance increases. For longer distances the price curve for transport by rail intersects with price for transport by water. The necessary and sufficient condition for achieving this ranking of intersections between the prices with respect to distance is derived in this article. The analysis addresses how the ranking depends on differences in both terminal costs and distance dependent marginal costs for the transport modes. It is demonstrated that the ranking with respect to distance can in fact be inversed. Knowledge of intersections between prices for transport modes can aid decision-makers when aiming to promote intermodal transport and achieve a more sustainable transport system.

Keywords: Freight transport policy; shippers’ preferences; transport mode; transport distance; transport price.

1. Introduction

The successful promotion of intermodal transport, using rail or sea on the long haul, has been identified as the most critical action to achieve a sustainable transport sector (Tsamboulas et al., 2007). Thus, intermodal transport is promoted through policies that are directed at all political levels (Macharis et al., 2011). However, there has been limited progress in shifting freight transport from roads to more efficient modes (European Commission, 2009), suggesting that the policies implemented thus far have not succeeded completely.

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1 Intermodal transport is the movement of goods in a single loading unit or vehicle that successively uses two or more modes of transport without the goods themselves being handled when changing modes (UN/ECE, 2001).
Knowledge about the factors determining the choice of transport services is a key to understanding the freight transport market and designing competitive transport systems (Flodén et al., 2010), and several studies have been conducted regarding transport service choices (for a review see Meixwell and Norbis, 2008). Cost is a factor of great importance to transport purchasers (shippers or forwarders), and is one of the most important attributes in the selection of a transport solution, particularly important for low value goods (e.g., Cullinane and Toy, 2000; Danielis and Marcucci, 2007; Punakivi and Hinkka, 2006).

For the movement of goods, shippers tend to prefer truck for shorter distances due its flexibility and the relatively low costs related to loading and unloading at the terminal (e.g. Rodrigue et al., 2009). Estimates suggest that compared to transport by water, the average freight cost per tonne-km is 3 times as high for transport by rail and 35 times as high for transport by truck (Ballou, 2004). Relative to truck, rail has higher costs related to handling at the terminal but involves lower costs for carrying a tonne an extra km. Hence, as the distance increases the advantage of lower distance-related costs will make rail preferable to truck at some point. Similarly, water transport is characterised by greater costs at terminals than rail (thereby also truck), but with considerably lower costs related to transporting the goods an extra km compared to rail and truck. Hence, when the distance increases, water transport will be preferred to rail at some point.

The aim of this article is to apply a model framework to discuss the reasonability of the ranking of the prices for transport modes stating that truck is preferred for short distances, followed by rail for longer distances, and, finally, water for the longest distances. It is demonstrated that this ranking of prices with respect to distance for different transport modes does not always take place even when accepting the basic assumptions of the ranking of terminal cost and the extra costs of transporting the goods an additional km. The model results can be used by policy-makers to evaluate in a better way the competitiveness of different modes of transport in the different market segments when working conditions (costs) change.

The further organisation of the paper is as follows. Section 2 presents a model explaining the relationship between the trip distance and the transport price for different transport modes. The necessary and sufficient condition for discussing the ranking of prices in relation to distance for transport modes are then presented in Section 3. The relevance of the model is demonstrated in Section 4 using data from the practice field. Finally, conclusions and implications are presented in Section 5.

2. The model – Cost, price and transport distance

The optimal transport solution depends on the objectives to be maximized. For a shipper this is usually to some degree related to minimizing the total transport costs. Total cost would include time cost and damage cost and is also referred to as generalized transport costs (see e.g. Button, 2010; Hanssen et al., 2012). In the following analysis it is, however, assumed that the shipper purely considers pecuniary costs.

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2 The shipper is defined as the producer of goods that determine the demand for transportation (Crainic and Laporte, 1997) and selected as the purchaser of transport services in the following analysis. Hence, we do not separate them from forwarders or brokers.
2.1 Cost functions

The costs of freight transport vary greatly between modes and type of goods. In general, it is reasonable that costs for the transport firm, \( C \), depend on the amount transported, \( X \), and the transport distance, \( D \). In equation (1) the influences of \( X \) and \( D \) on \( C \) are represented by a linear relationship, which is an example of a simple cost function with the advantage of being able to draw simple interpretations. It implies that costs increase linearly with tonnes, \( X \), and tonne km, \( XD \). Despite the weakness of treating all transport services as a homogenous product, common output measures are tonne and/or tonne km (e.g. Coelli et al., 2005; Pels and Rietveld, 2008). More advanced cost functions could also be used to capture more of the variation in costs, but several empirical studies indicate that simple cost functions can be good proxies for more advanced functions, see for example Pels and Rietveld (2008).

\[
(1) \quad C = \alpha_0 + \alpha_1 X + \alpha_2 XD \quad \text{where} \quad \alpha_0, \alpha_1, \alpha_2 > 0 \quad \text{and} \quad \frac{\partial C}{\partial X} = \alpha_1 + \alpha_2 D
\]

In (1) costs that are independent of the size of the shipment and distance are indicated by the parameter \( \alpha_0 \). Marginal costs, \( \frac{\partial C}{\partial X} = \alpha_1 + \alpha_2 D \), in this simple cost function increase linearly with transport distance. Thus, \( \alpha_1 \) in (1) can be interpreted as distance-independent marginal costs while \( \alpha_2 \) indicates the cost of transporting one tonne an additional km.

The assumptions about the cost structure regarding freight transportation, as previously mentioned in Section 1, suggest that \( \alpha_1 \) is lowest (highest) for truck (water) transport while \( \alpha_2 \) is lowest (highest) for water (truck) transport. The cost structure for rail lies in between these two extremes.

2.2 Cost functions and transport prices

Costs make an important basis for forming of prices in the transport industry. Under perfect competition prices reflect marginal cost (e.g. Carlton and Perloff, 2005). It is a major simplification, but three factors indicate that assuming perfect competition might not be an unreasonable approach when analysing the market for freight transport in general. First, the setting of prices for freight transport is less standardised and regulated by the authorities than public passenger transport (Blauwens et al., 2008). The prices are, broadly speaking, set freely in the market. Second, freight transport firms are generally profit maximising entities. Finally, barriers to entry and exit are low. Essentially, firms with available capacity related to trucks or vessels can enter new markets for freight transport at will. This suggests that the relationships between fare and distance for the different modes are approximately equal to their marginal costs, \( (\alpha_3 + \alpha_2 D) \), as defined in (1).

When we assume a highly competitive market and the cost structure for water transport, rail transport, and truck transport described above, the relationship between price (equal to marginal costs) and distance is defined in (2) for transport by truck, rail or water using the subscripts \( t, r \) and \( w \), respectively.

\[
(2) \quad P_t = \beta_0 + \beta_1 D \quad \text{(truck)}, \quad P_r = a\beta_0 + c\beta_1 D \quad \text{(rail)} \quad \text{and} \quad P_w = ab\beta_0 + cd\beta_1 D \quad \text{(water)}
\]

where \( \beta_0, \beta_1 > 0, a, b > 1 \) and \( 0 < c, d < 1 \).
Prices for truck, rail and water transport are in (2) denoted as $P_t$, $P_r$ and $P_w$, respectively. The price of truck, $P_t$, forms the basis. The constant and increasing slope of the price curve for truck are defined by $\beta_0$ and $\beta_1$, respectively. $\beta_0$ represents distance-independent price for services such as loading and unloading while $\beta_1$ shows how price increases with distance. The distance dependent price is, thus, $(\beta_1 D)$. The distance-independent costs for rail relative to truck and for water relative to rail are represented by parameters $a$ and $b$, respectively. Similarly, the relative magnitudes of distance-dependent costs are represented by parameters $c$ and $d$ for rail and water, respectively.

The restrictions on the parameters $a, b, c$ and $d$ ensure that $ab\beta_0 > a\beta_0 > \beta_0$ and $\beta_1 > c\beta_1 > cd\beta_1$, meaning that distance-independent transport cost (terminal cost) are highest for water transport and lowest for truck transport, whereas the costs of transporting one unit an extra kilometre are highest for truck transport and lowest for water transport $(\partial P_t/\partial D > \partial P_r/\partial D > \partial P_w/\partial D)$. Moreover, the price per km $(P_t/D)$, decreases with distance $(\partial (P_t/D)/\partial D < 0)$ for all transport modes. When the distance approaches infinity, the price per km, approaches the distance dependent element.

If, for example, rail has 10% higher terminal costs and 10% lower distance-related costs relative to truck, then $a = 1.1$ and $c = 0.9$. Moreover, if water has 10% higher terminal costs and 10% lower distance-related costs relative to rail, then $b = 1.1$ and $d = 0.9$. In this case water has 21% higher terminal costs $(ab = 1.1 \cdot 1.1 = 1.21)$ and 19% lower distance-related costs $(1 - 0.9 \cdot 0.9 = 0.19)$ relative to truck.

2.3 Threshold distances

The threshold distances for when one transport mode is preferred to another for a shipper who wishes to minimize transport price can be derived from (2). Equation (3) presents the threshold distances (upper limit) ensuring that the price for transport by truck is lower than transport by rail, $D_{tr}$, transport by truck is lower than transport by water, $D_{tw}$, and transport by rail is lower than transport by water, $D_{rw}$.

\[
\begin{align*}
(3) \quad P_t < P_r & \Rightarrow D < D_{tr} = \frac{\beta_0 a^{-1}}{\beta_1 1-c} \\
P_t < P_w & \Rightarrow D < D_{tw} = \frac{\beta_0 ab^{-1}}{\beta_1 1-cd} \\
P_r < P_w & \Rightarrow D < D_{rw} = \frac{\beta_0 a(b^{-1})}{\beta_1 c(1-d)}
\end{align*}
\]

The distances at which the relationships between prices and distance intersect, as defined by $D_{tr}$, $D_{tw}$ and $D_{rw}$, will all be positive provided that the presumed ranking of parameters is satisfied. It can easily be deduced from (3) that all intersection distances increase with the difference between the distance-independent elements $(a\beta_0, a\beta_0, \beta_0)$ and decrease with the difference between the marginal increases in price with respect to transport distance $(cd\beta_1, cd, \beta_1)$. A closer study of the differences between the intersections defined in (3) shows that the partial differentiations with respect to $b$ provide unambiguous results. Hence, a higher value of $b$ will increase the differences $(D_{tw} - D_{tr})$, $(D_{rw} - D_{tr})$ and $(D_{rw} - D_{tw})$.

The common ranking of the threshold distances is that $D_{tr} < D_{tw} < D_{rw}$ which means that the price for transport by truck is lowest for distances less than $D_{tr}$, rail is lowest for distances between $D_{tr}$ and $D_{rw}$ and water is lowest for distances above $D_{rw}$ (see e.g. Rodrigue et al., 2009). This ranking is illustrated in Figure 1 using the notation from (2).
Even when accepting the bindings on the parameters imposed in (2), the ranking of the threshold distances indicated in Figure 1 making the different modes most competitive, as far as prices are concerned, cannot be regarded as a general result. We will discuss this problem more thoroughly in the next section.

3. Conditions for ranking of the threshold distances

3.1 Necessary and sufficient conditions for the commonly accepted ranking of threshold distances

Since $\beta_0$ and $\beta_1$ are positive, the expressions in (3) can be multiplied by $\beta_1/\beta_0$ making $D_{tr} < D_{tw} < D_{rw}$ equivalent to

$$\frac{a-1}{1-c} < \frac{ab-1}{1-cd} < \frac{a\left(b-1\right)}{c\left(1-d\right)}.$$ 

Both the left inequality, $\frac{a-1}{1-c} < \frac{ab-1}{1-cd}$, and the right inequality, $\frac{ab-1}{1-cd} < \frac{a\left(b-1\right)}{c\left(1-d\right)}$, can be rephrased as an expression by the parameter $b$ as demonstrated in (4) (see appendix). Hence, equation (4) gives the necessary and sufficient conditions for ensuring the ranking of threshold distances illustrated in Figure 1.  

$$b > \frac{a-1}{a} \frac{c\left(1-d\right)}{1-c} + 1 \text{ where } a > 1 \text{ and } 0 < c, d < 1$$

Moreover, the sufficient condition for the same ranking is:

$$b > \frac{c\left(1-d\right)}{1-c} + 1 \text{ where } 0 < c, d < 1$$

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3 It is worth noting that the relationship between price and travel distance can be made more general by replacing $D$ in (2) by $D^\tau$ where $\tau > 0$. If $\tau$ is equal for all transport modes, the conditions in (4) and (5) are still valid. When $\tau > 1$, $\tau = 1$, $\tau < 1$ then prices increase convexly, linearly and concavely with distance, respectively.
A number of conclusions can be drawn from (4) and (5). First, the ranking is independent of the variables $\beta_0$ and $\beta_1$. This is reasonable since both parameters are included in the expressions for price for all transport modes and do not influence the relative relationship between their costs. Second, the fact that the conditions can be solved and easily expressed by the parameter $b$, shows that the common ranking is achieved as long as the constant term of transport by water, in which $b$ is included, is large enough. A higher value of $b$ will shift $P_w$ upwards and move the intersection with $P_t$ to higher distances. Third, partial differentiations of the right hand side of the inequality in (4) are positive with respect to $a$ and $c$ and negative with respect to $d$. Hence, a higher constant for rail, $a$, and a more steeply increasing slope for rail, $c$, requires a higher value of $b$. Oppositely, a more steeply increasing slope for water, $d$, reduces the threshold value of $b$. Fourth, when $a$ increases, $(a - 1)/a$ in (4) moves towards 1. Hence, the sufficient and necessarily conditions for the above ranking become more equal if the distance-independent prices for rail and water transport increase relative to truck transport. Finally, it follows from (5) that the sufficient condition for the commonly accepted ranking is independent of the relative magnitude of distance independent prices for rail and truck ($a$ value).

3.2 Alternative ranking of threshold distances

If the right and left side in (4) are equal, then a situation arises in which all three transport modes have the same price at a given distance. This is illustrated in Figure 2 by the downward shift of curve $P_w$ to $P_w^{**}$ by a reduction in $b$. This value of $b$ gives an intersection of the curve $P_w^{**}$ with $P_t$ and $P_r$ in distance $D_{tr}$.
If $b$ is further reduced, but still not in contradiction to the parameter restriction $b > 1$ defined in (2), then the inequality sign of (4) is turned.\textsuperscript{4} This gives a situation where the ranking of intersections is the opposite of that illustrated in Figure 1. This is also demonstrated in Figure 2 by a further downward shift of curve $P_w$ to $P_w^*$. The new intersections for water transport with truck and rail are denoted by $D_{tw}^*$ and $D_{rw}^*$, respectively. Hence, the ranking is then $D_{rw}^* < D_{tw}^* < D_{tr}$. Rail is, however, not preferred for any distances since truck has the lowest price for distances shorter than $D_{tw}^*$ and water is preferred for longer distances. The relationship between the lowest transport price and distance is indicated by the bold line in Figure 2.

3.3 Numerical examples

The relationship between the parameters in the necessary and sufficient condition ensuring the ranking of the threshold distances illustrated in Figure 1 is shown in Table 1. In the sensitivity analyses presented in Table 1 the parameter values are indicated by three scenarios representing a 10%, 20% and 30% change in accordance with the parameter restrictions. For the values $c$ and $d$ these scenarios represent the parameter values 0.9, 0.8 and 0.7 and for $a$ these values are 1.1, 1.2 and 1.3.

<table>
<thead>
<tr>
<th>Required value of $b$</th>
<th>$d$ when $a = 1.1$</th>
<th>$d$ when $a = 1.3$</th>
<th>$d$ when $a = 1.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.9  0.8  0.7</td>
<td>0.9  0.8  0.7</td>
<td>0.9  0.8  0.7</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9  1.08  1.25</td>
<td>1.15  1.30  1.45</td>
<td>1.21  1.42  1.62</td>
</tr>
<tr>
<td></td>
<td>0.8  1.04  1.11</td>
<td>1.07  1.13  1.20</td>
<td>1.09  1.18  1.28</td>
</tr>
<tr>
<td></td>
<td>0.7  1.02  1.06</td>
<td>1.04  1.08  1.12</td>
<td>1.05  1.11  1.16</td>
</tr>
</tbody>
</table>

As discussed in relation to equation (4) it is evident also from Table 1 that the required value of $b$ increases with higher values of $a$ and $c$ and lower values of $d$. If, for example, the distance-independent price for rail transport is 20% higher than for truck transport ($a = 1.2$), the increase in price when transporting the goods en extra km is 20% lower for rail than for truck transport ($c = 0.8$) and 10% lower for water transport than for rail transport ($d = 0.9$), then the distance-independent price for water transport must be 7% higher than for rail transport ($b = 1.07$). The results from Table 1 for $a = 1.2$ are visualized in Figure 3 using the full interval for the parameters. All $b$-values above the surfaces ensure that the inequality in (4) is fulfilled for the different values of $a$, $c$ and $d$.

\textsuperscript{4} It can be seen from (4) that this inversed ranking takes place if $1 < b < 1 + \frac{a - 1 + c (1 - d)}{a}$. If, for example, $a$ is only marginally higher than 1 then the element $(a - 1)/a$, will be small and there is limited possibilities for variation in the parameter $b$ giving this solution.
Figure 3: Required value of $b$ for different values of $c$ and $d$ ensuring the suggested ranking of threshold distances when $a = 1.2$.

It is demonstrated in Table 1 and Figure 3 that the relative difference between the increasing slope of truck and rail, represented by $c$, is more influential on the threshold value of $b$ than the difference between rail and water, represented by $d$. The effect of $d$ is further reduced with lower values of $a$ and $c$. Moreover, the influence of $c$ diminishes rapidly as the value moves from 1 towards 0. Hence, if $c$ is just below 1, meaning that the distance dependent costs of rail are only slightly lower than that of truck, then $b$ must be very high in order to obtain the common ranking in Figure 1.

The sufficient threshold value of $b$ resulting in the common ranking of threshold distances (making inequality (5) fulfilled) for different values of $b$ and $c$ is illustrated in Figure 4. Note that the scale for the $b$-value is different in Figure 4 as compared to Figure 3. As expected, the sufficient threshold values of $b$ are higher than the necessary ones. It is also evident that both $c$ and $d$ have greater influence on the sufficient value of $b$ than on the necessary value of $b$. 
Figure 4: The required value of \( \varepsilon \) for the sufficient condition giving parameter combinations of \( c \) and \( d \) that ensure the suggested ranking of threshold distances.

4. An example from practice

In practice the values of the parameters will vary according to the characteristics of the goods transported. Any general definition of values is therefore uncertain; in particular the relative magnitudes of the distance-independent costs for truck, rail and water transport measured by the parameter values of \( a \) and \( b \). From different data sources (Kim and Van Wee, 2011; Norlins, 2012) we are, however, able to give rough estimates of the extra costs of transporting a container an additional kilometre for the three modes in question. These costs were 30% lower for rail than for truck and 4% lower for water transport than for rail. This implies that \( c \approx 0.70 \) and \( d \approx 0.96 \).

The relationship between the distance-independent elements ensuring the necessary and sufficient conditions in (4) and (5) respectively are fulfilled, can be deduced by the use of the derived values \( c = 0.70 \) and \( d = 0.96 \). The area above the concave curve in Figure 5 gives the ranking \( D_{tr} < D_{tw} < D_{rw} \), while the area below the curve gives the ranking \( D_{rw} < D_{tw} < D_{tr} \) as discussed in relation to Figure 2. If, for example the value of \( a \) is 1.2 and 1.5, the necessary values of \( b \) ensuring the common ranking of threshold distances, \( D_{tr} < D_{tw} < D_{rw} \), are 1.02 and 1.03, respectively. Figure 5 shows that the

\[ 5 \text{ This means that costs related to transporting a container an extra kilometer by water is about 33% lower than for truck (}1 - 0.70 \cdot 0.96\).]
relationship between $b$ and $a$ increases concavely implying that the higher the value of $a$ the lower its marginal influence on the threshold value of $b$.

Figure 5: Relationship between threshold values of $b$ and $a$ when $c = 0.70$ and $d = 0.96$.

The horizontal dotted line in Figure 5 illustrates the threshold value of $b = 1.09$ according to the sufficient condition. It represents the asymptotic value of $b$ when $a \to \infty$. Hence, for reasonable values of the relative magnitudes of the extra costs of transporting goods an additional kilometre for different modes (the derived values of $c$ and $d$), the common ranking of threshold distances ($D_{tr} < D_{tw} < D_{rw}$) is fulfilled when the terminal costs for water transport is 9% higher than for rail transport, regardless of the relative magnitude of the distance-independent transport costs of rail as compared to truck transport (value of $a$).

5. Conclusions and implications

A model has been established as a framework for discussing the ranking of prices for transport by truck, rail and water according to transport distance. It is suggested that the distance-independent price (e.g. cost for loading and unloading at terminal) is lowest for truck, followed by rail and highest for water, while the extra costs of transporting the goods an extra kilometre is highest for truck, followed by rail and lowest for water. As a result, when aiming to minimize transport price shippers tend to prefer truck for short distances, rail for medium distances, and water transport for the longest distances. However, in this paper it is argued that the ranking of prices with respect to distance does not necessarily follow from the assumptions regarding the rankings among the modes of terminal cost and extra costs of transporting the goods an additional kilometre.
The necessary and sufficient conditions for obtaining their ranking of modes with respect to distance can be expressed by the parameters indicating the relative magnitudes of distance-independent costs (parameters $a$ and $b$) and distance dependent costs (parameters $c$ and $d$) for the three modes in question. It is shown that the necessary and sufficient condition for this ranking can be solved and expressed by the parameter indicating how much higher the distance-independent element of water is relative to rail (parameter $b$). A higher value of parameter $b$ will shift the curve representing the price for transport by water, $P_w$, upwards and move the intersection with the price curve for rail, $P_r$, to higher trip distances. Oppositely, it is also shown that if the distance-independent cost of transport by water is reduced sufficiently relative to rail transport, then the opposite ranking of intersections will occur. In this case rail will not be the preferred alternative for any distance since road transport still has the lowest cost for short distances.

A further study of the necessary and sufficient condition shows that a higher terminal cost for rail ($a$) and a more steeply increasing slope with distance for rail ($c$), requires a higher terminal cost of water to obtain the common accepted ranking. Oppositely, a more steeply increasing slope with distance for water compared to rail ($d$) reduces the value of $b$ required to obtain the common ranking. However, the value of $c$ is much more important than $d$ for achieving the previously mentioned ranking. For reasonable values of $c$ and $d$ it is derived that the common ranking is achieved for all values of $a$ if the terminal costs of water transport are more than 9% higher than for rail transport ($b > 1.09$).

It is demonstrated how the relative magnitudes of distance-independent and distance dependent transport costs, represented by the parameters $a, b, c$ and $d$ influence shippers’ choice of transport modes over different distances. It is worth noting that low terminal costs for water transport relative to rail ($b$ marginally higher than 1) may imply that rail transport is not preferred for any distances.

Transport authorities and regulators can influence the values of $a, b, c$ and $d$. Consequently, our results are useful for informing decision-makers with regard to designing a more sustainable transport system. If, for example, investments are made at ports and intermodal terminals to improve handling efficiency, then the distance-independent elements ($a$ and $b$) will be relatively less influential. This means that the advantages of lower distance dependent costs for rail and water relative to truck will be more prominent and the threshold distance for preferring rail and water transport rather than truck will be reduced. If for example environmental taxes are imposed on use of truck, as suggested by the greening package for Europe (European Commission, 2008), then $c$ and $d$ will be further reduced (the difference increases) and reduce the attractiveness of truck.

Finally, we find it important to outline the two most critical assumptions of the article and what objections they give rise to. Firstly, the model assumes that shippers choose transport modes based on pecuniary transport costs alone. For high-value goods and fresh products quality factors such as transport time and reliability may count more than price. Secondly, the relationships between pecuniary transport costs for truck, rail and water on one hand and transport distance on the other hand are based on (1) perfect competitions in the freight market and (2) linear relationships between transport firms’ costs and transport distance. Even though these are reasonable assumptions in general, there are numerous of examples of market segments where there are only one or very few suppliers.
Despite these limitations, the paper has nevertheless established a model to analyse thoroughly how the competitiveness of truck, rail and water transport over different transport distances varies with the modes’ cost structure.

References


Appendix

Theorem 1. Let $D_{tr}$ be the distance at which $P_r$ and $P_t$ intersect, let $D_{tw}$ be the distance at which $P_r$ and $P_w$ intersect, and let $D_{rw}$ be the distance at which $P_r$ and $P_w$ intersect. Suppose that $a, b > 1$ and $0 < c, d < 1$. Then

(I) $D_{tr} < D_{tw} < D_{rw}$

is equivalent to

(II) $b > 1 + \frac{a^{-1} c (1 - d)}{a \cdot \frac{1 - c}{1 - c} - d}$.

Also, $b > 1 + \frac{c}{1 - c} \cdot (1 - d)$ is a sufficient condition for the order of the intersections given by (I).

Proof: The distances at which the curves intersect are given by $D_{tr} = \frac{\beta_a a^{-1}}{\beta_1 \cdot c (1 - d)}$, $D_{tw} = \frac{\beta_b ab^{-1}}{\beta_1 \cdot 1 - cd}$, and $D_{rw} = \frac{\beta_a a(b-1)}{\beta_1 \cdot c (1 - d)}$. Thus, we get the following sequence of equivalences $D_{tr} < D_{tw} < D_{rw}$.

Using the formulas above gives $\frac{\beta_a a^{-1}}{\beta_1 \cdot c (1 - d)} < \frac{\beta_b ab^{-1}}{\beta_1 \cdot 1 - cd} < \frac{\beta_a a(b-1)}{\beta_1 \cdot c (1 - d)}$.

Multiplying by $\frac{\beta_1}{\beta_1}$, which is positive, gives

(III) $\frac{a^{-1}}{1 - c} < \frac{ab^{-1} - c}{1 - cd} < \frac{a(b-1)}{c (1 - d)}$.

Next, we show that both the left and the right inequality of (III) are equivalent to (II). The idea is to solve these inequalities with respect to the quantity $(b - 1)$ because the right side of (II) is simplified when subtracting 1.

We start with the left inequality of (***):$\frac{a^{-1}}{1 - c} < \frac{ab^{-1} - c}{1 - cd}$

Substituting $b$ by $b - 1 + 1$ gives $\frac{a^{-1}}{1 - c} < \frac{a(b-1) - a}{1 - cd}$. Multiplying by $(1 - c)(1 - cd) > 0$ gives $(a - 1)(1 - cd) < a(1 - c)(b - 1) + (a - 1)(1 - c)$.

Moving terms gives $-a(1 - c)(b - 1) < (a - 1)(-1 + cd) + (a - 1)(1 - c)$.

Multiplying by $-1$ and simplifying gives $a(1 - c)(b - 1) > (a - 1)(c - cd)$.

Dividing by $a(1 - c)$ gives $b - 1 > \frac{a}{1 - c} \cdot \frac{c}{1 - c} \cdot (1 - d)$.

Adding 1 to each side gives $b > 1 + \frac{a}{a \cdot \frac{1 - c}{1 - c} \cdot (1 - d)}$ which is the same as (II).

Now, let us show that the right inequality of (III) is equivalent to (II).

We start with the right inequality of (***): $\frac{ab^{-1}}{1 - cd} < \frac{a(b-1)}{c (1 - d)}$

Replacing $b$ on the left side by $(b - 1 + 1)$ gives $\frac{a(b-1) + a}{1 - cd} < \frac{a(b-1)}{c (1 - d)}$.

Multiplying by $(1 - cd)c(1 - d)$ gives $ac(1 - d)(b - 1) + (a - 1)c(1 - d) < a(1 - cd)(b - 1)$.
Simplifying the coefficients of the \( (b - 1) \)-terms gives \( (ac - acd)(b - 1) + (a - 1)c(1 - d) < (a - acd)(b - 1) \).
Adding the \( (b - 1) \)-terms gives \( (ac - acd - a + acd)(b - 1) < -(a - 1)c(1 - d) \).
Simplifying and multiplying by \(-1\) gives \( (a - ac)(b - 1) > (a - 1)c(1 - d) \).
Dividing by \( a - ac = a(1 - c) \) gives \( b - 1 > \frac{a - 1}{a} \cdot \frac{c}{1 - c} \cdot (1 - d) \).
Adding 1 to each side gives \( b > 1 + \frac{a - 1}{a} \cdot \frac{c}{1 - c} \cdot (1 - d) \) which is the same as (II).

To prove the last statement in Theorem 1, note that when \( a > 1 \) and \( 0 < c, d < 1 \), then \( 0 < \frac{a - 1}{a} < 1 \) and \( \frac{c}{1 - c} \cdot (1 - d) > 0 \). Hence, \( 1 + \frac{c}{1 - c} \cdot (1 - d) > 1 + \frac{a - 1}{a} \cdot \frac{c}{1 - c} \cdot (1 - d) \).
Consequently, \( b > 1 + \frac{c}{1 - c} \cdot (1 - d) \) implies that \( b > 1 + \frac{a - 1}{a} \cdot \frac{c}{1 - c} \cdot (1 - d) \) which is equivalent to (I). Thus, (II) is a sufficient condition for the order of the intersections given by (I).