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New Thoughts about an Old Result on Univalent Functions
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Haakon Waadeland

New Thoughts about an Old Result on Univalent Functions
New Thought in Univalent Functions

HAAKON WAADELAND

1. Prologue.

In the theory of the analytic functions which have attributes according to G. F. Julia's model of classification, one such area is the study of the behavior of functions that deal with functions of the form $f(z)$ in the unit disk $U$, and in particular, around $z = 0$ the

and converges in the domain $|z| < 2$ and (in a footnote it is claimed) really "took off" after, greatly helping the Leningrad group with the conjecture [2].

Few things are as important (in particular if one considers a large and great number of theorems, coefficients, generalizations, and created, methods, etc.) have become clear that parameter methods with two variational aspects are the most results having individually meaning scope, being included.

The topic of this book has no value in itself, but interestingly, and hopefully...
New Thoughts about an Old Result on Univalent Functions

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1. Prologue.

In the theory of functions of a complex variable there are certain areas which have attracted and challenged a large number of mathematicians. One such area is the theory of univalent functions (or "schlicht" functions, according to German tradition). Restricted to the most known case it deals with functions $f$, analytic and univalent (one to one) in the open unit disk $U$, and normalized by $f(0) = 0, f'(0) = 1$. The Taylor expansion around $z = 0$ thus has the form

$$w = f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$  \hspace{1cm} (1.1)

and converges in $U$. When Ludwig Bieberbach in 1916 proved that $|a_2| \leq 2$ and (in a foot-note) conjectured that generally $|a_n| \leq n$ [1] the theory really "took off", and it did not even stop when de Branges 68 years after, greatly helped by Richard Askey, Walter Gautschi and, above all, a Leningrad group of mathematicians, proved the truth of the Bieberbach conjecture [2].

Few things are more useful in mathematics than difficult conjectures (in particular if they have a simple appearence). Throughout the years a large and great theory has been developed, with growth and distortion theorems, coefficient results of different types, results for subfamilies, for generalizations etc. Most important perhaps, is that methods have been created, methods of value far beyond the original intention. Some results have become classical parts of the theory, such as for instance Löwner's parameter method, inspired by thoughts in hydrodynamics, and Schiffer's two variational methods (together with other variational methods). But most results have been small, modest contributions to the theory, individually meaning very little, but together, in virtue of multitude and wide scope, being indispensable in the theory.

The topic of the present paper is such a small result, being of little or no value in itself, but of some pleasure and joy for the one who created it, and hopefully also to the ones who have read it. The result was, at
least indirectly, inspired by Ernst Jacobsthal, who once told a young assistant about univalent functions and in particular about the Bieberbach conjecture. The assistant readily started off to prove it. After several attempts and a lot of reading in books and papers, he realized that it was not all that simple, and put it aside (for a while). He still, however, maintained his interest in the field. One day he asked himself the question: "What is the effect on the second coefficient \( a_2 \) to require \( a_3 = 0 \)?" The answer was written down, then supported by Jacobsthal, and (surprisingly) approved of by Bieberbach, and turned out to be the very first paper in this field by the young mathematician [10].

43 years later, when the young mathematician had turned into an old mathematician, he looked again at his paper. He got some new thoughts, where things that had happened in the field in the meantime were taken into account. The purpose of the present paper is to tell about these thoughts. But first we need to take a look at the old result.

2. The old result.

Let \( S \) denote the family of "normalized, univalent functions", i.e. the family of functions presented in the introduction, and where we have the expansion

\[
w = f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_n z^n + \cdots. \tag{2.1}
\]

Let \( S' \) be the subfamily of \( S \), defined by the additional requirement \( a_3 = 0 \), and hence with the expansion

\[
w = f(z) = z + a_2 z^2 + a_4 z^4 + \cdots + a_n z^n + \cdots. \tag{2.1'}
\]

Two questions were asked: a) What is the best upper bound for \( |a_2| \) in \( S' \)? b) What is the best (largest) number \( d \), such that the disk \( |w| \leq d \) is contained in \( f(U) \) for all \( f \in S' \)?

Remark.

If the question b) had been asked for the class \( S \) instead of \( S' \), the answer had been \( 1/4 \), the Koepke constant. Paul Koepke proved, already in 1907, the existence of such a positive constant [5]. The actual value \( 1/4 \) was proved by Bieberbach. The functions (the only ones) for which the nearest boundary point of \( f(U) \) has the distance \( 1/4 \) from the origin are the functions

\[
w = k(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots \tag{2.2}
\]

and its rotations, i.e. the functions \( k_{\phi}(z) := e^{-i\phi} k(e^{i\phi} z) \). These functions are also the only ones for which \( |a_2| = 2 \). The function (2.2) is called the Koepke function and is the largest constant \( \kappa \) such that

\[
f(U), \, f \in S'.
\]

We shall not bore ourselves by means of the previous results, but have, that for a family of functions \( S' \) with first coefficients

\[
a_2 = 1, \quad a_3 = 0, \quad a_4 = \kappa(1 - 2a_2), \quad \ldots,
\]

where \( \kappa(\tau) \) is positive and

The bounds are given by

and its rotations, i.e., the functions \( k_{\phi}(z) := e^{-i\phi} k(e^{i\phi} z) \). These functions are also the only ones for which \( |a_2| = 2 \). The function (2.2) is
called the Koebe function and maps $U$ onto the whole plane, minus a slit from $-1/4$ to $\infty$ along the negative real axis. A solution to our question b) could then be called a "Koebe result", and the best $d$-value could be called a "Koebe constant" for the family $S'$.

The results in the paper [10] are:

$$|a_2| \leq 1. \quad (2.3)$$

The disk

$$|w| \leq \frac{1}{3} \quad (2.4)$$

is the largest disk, centered at the origin, that is contained in all sets $f(U)$, $f \in S'$.

We shall not include the proof here, only mention that it was established by means of the Löwner method, where (among many other things) we have, that for a dense subfamily of $S$ the following formulas hold for the first coefficients:

$$a_2 = -2 \int_0^\infty \kappa(\tau)e^{-\tau}d\tau, \quad (2.5)$$

$$a_3 = 4\left(\int_0^\infty \kappa(\tau)e^{-\tau}d\tau\right)^2 - 2 \int_0^\infty \kappa^2(\tau)e^{-2\tau}d\tau, \quad (2.6)$$

where $\kappa(\tau)$ is piecewise continuous and of absolute value 1, $|\kappa(\tau)| = 1$ [7].

The bounds (2.3) and (2.4) are best possible. The extremal functions (and the only ones) are

$$w = f(z) = \frac{z}{1 - z + z^2} \quad (2.7)$$

and its rotations. The function (2.7) maps the unit disk onto the whole plane minus two slits, one going from $-1/3$ along the negative real axis to $-\infty$, the other one going from 1 to $\infty$ along the positive real axis.

In the paper [10] are also studied $k$-symmetric functions in $S$ with vanishing third coefficient, i.e. functions

$$w = g_k(z) = z + a_{k+1}^{(k)}z^{k+1} + a_{2k+1}^{(k)}z^{2k+1} + a_{3k+1}^{(k)}z^{3k+1} + \cdots, \quad (2.8)$$

where $a_{2k+1}^{(k)} = 0$. The following sharp bound was found:

$$|a_{k+1}^{(k)}| \leq \sqrt{\frac{2}{k(k+1)}} \quad (2.9)$$
Moreover, the "Koebe constant" for this family was proved to be

$$ \left( 2 + \sqrt{\frac{2k}{k+1}} \right)^{-1/k} \quad (2.10) $$

The only extremal functions are

$$ w = g_k(z) = \frac{z}{\left( 1 - \sqrt{\frac{2k}{k+1}} z^k + z^{2k} \right)^{1/k}} \quad (2.11) $$

and its rotations.

So far the old result. In the following we shall restrict ourselves to the case $k = 1$, and we shall not be aiming at any "Koebe result".

3. The new thoughts.

Let $M$ be an arbitrary number in $[1, \infty]$. An interesting subfamily of $S$ is the one obtained by the additional condition

$$ |f(z)| < M \quad (3.1) $$

for all $z \in U$. This family is usually denoted $S_M$. Two trivial cases: For $M = 1$ the family consists of merely one function, $f(z) = z$, whereas $S_\infty = S$. From now on we shall assume that $1 < M < \infty$.

The different questions dealt with in the class $S$ are in most cases substantially more difficult to handle in $S_M$. For the first two coefficients the sharp bounds in $S_M$ are:

$$ |a_2| \leq 2(1 - \frac{1}{M}), \quad [8] \quad (3.2) $$

$$ |a_3| \leq 1 - \frac{1}{M^2} \quad \text{for} \quad M \leq e, \quad (3.3a) $$

$$ |a_3| \leq 2e^{-2\nu} - \frac{4e^{-\nu}}{M} + 1 + \frac{1}{M^2} \quad \text{for} \quad M \geq e, \quad (3.3b) $$

where $\nu \in [0, 1]$ is given by $M^{-1} = \nu e^{-\nu}$. See e. g. [9], and also [4, p. 54].

Apart from rotations there is only one function in $S_M$ with equality in (3.2), namely the function

$$ w = Mk^{-1} \left( \frac{k(z)}{M} \right) = z + 2(1 - \frac{1}{M})z^2 + (1 - \frac{1}{M})(3 - \frac{5}{M})z^3 + \cdots \quad (3.4) $$

and

$$ a_3 = 4 \left( \frac{1}{M} \right) \quad (3.5) $$

To simplify we have

$$ k(z) = Mz \quad (3.6) $$

and

$$ \nu \in [0, 1] \quad (3.7) $$

Membership in $S_M$.

(This form is correct.

The equation obtained by taking the upper bound for $|a_3|$ in $S_M$.

In spite of the first term on the hand side of (3.7)
There are some scattered results under additional conditions for higher order coefficients, but a sharp upper bound for $|a_n|$ generally seems to be located way beyond mountains of difficulties.

In the present section we shall determine the sharp upper bound for $|a_2|$ in the family $S'_M$, consisting of all $f \in S_M$ for which the coefficient $a_3 = 0$. The non-triviality we will experience in this investigation of the very first, presumably simple coefficient, will throw some light on the impact of the two conditions $|f(z)| < M$ and $a_3 = 0$ on the complexity of problems in the family of functions.

As in the old problem we shall also here use the Löwner formulas, which, however, differ from the formulas (2.5) and (2.6) by having $\log M$ as the upper limit instead of $\infty$, $\log$ being the natural logarithm. We thus have

$$a_2 = -2 \int_0^{\log M} \kappa(\tau)e^{-\tau}d\tau$$

and

$$a_3 = 4 \left( \int_0^{\log M} \kappa(\tau)e^{-\tau}d\tau \right)^2 - 2 \int_0^{\log M} \kappa^2(\tau)e^{-2\tau}d\tau .$$

To simplify writing we define

$$I_1 := \int_0^{\log M} \kappa(\tau)e^{-\tau}d\tau$$

and

$$I_2 := \int_0^{\log M} \kappa^2(\tau)e^{-2\tau}d\tau .$$

Membership in $S'_M$ is then equivalent to the relation

$$4I_2^2 = 2I_1 .$$

(This form is convenient, since $a_2^2 = 4I_2^2$.)

The equation (3.9) produces right away a trivial upper bound for $|a_2|$, obtained by taking $\kappa(\tau)$ to be 1, which gives us $1 - 1/M^2$ as the sharp upper bound for the right-hand side of (3.9), from which we get

$$|a_2| \leq \sqrt{1 - \frac{1}{M^2}} .$$

In spite of the fact that this bound comes from a sharp bound of the right-hand side of (3.9) it does not have to be sharp, the reason being that we
may not always have the possibility of the combination of equality in (3.9) and maximality on the right-hand side. We can immediately find a case when the bound (3.10) is not sharp, by looking at the sharp bound (3.2) for $|a_2|$ in $S_M$. Since $S'_M \subset S_M$, the sharp upper bound for $|a_2|$ in $S'_M$ must be less than or equal to the bound in $S_M$. Hence, if

$$\sqrt{1 - \frac{1}{M^2}} > 2(1 - \frac{1}{M})$$

which means that $M < \frac{5}{3}$, the bound (3.10) is not sharp. This raises two questions: a) Is the bound sharp for $M \geq \frac{5}{3}$? b) How can we find a sharp upper bound in the case $M < \frac{5}{3}$? We shall see that the answer to the question a) is yes:

**Proposition 1.**

For $M \geq \frac{5}{3}$ the following sharp bound holds in $S'_M$:

$$|a_2| \leq \sqrt{1 - \frac{1}{M^2}}. \quad (3.10)$$

**Proof:**

Take first $M = \frac{5}{3}$. In this case the extremal functions in $S_M$ are in $S'_M$, as seen in (3.4), and the two $|a_2|$-bounds are equal. Hence the statement in Proposition 1 is established for $M = \frac{5}{3}$.

Next take $M > \frac{5}{3}$, in this case

$$\sqrt{1 - \frac{1}{M^2}} < 2(1 - \frac{1}{M}). \quad (3.11)$$

In order to prove sharpness we need to find a $\kappa(\tau)$, such that $|2I_1| = \sqrt{1 - 1/M^2}$. For any $t_0 \in [0, \log M]$ take $\kappa(\tau)$ to be 1 for $\tau \in [0, t_0]$ and -1 for $\tau \in (t_0, \log M]$. Here we have $2I_2 = 1 - 1/M^2$, regardless of the $t_0$-value. When $t_0$ increases from 0 to $\log M$, the expression $-2I_1$ decreases from $2(1 - 1/M)$ to $-2(1 - 1/M)$. Since (3.11) holds, and since $-2I_1$ is a continuous function of $t_0$, there are two $t_0$-values for which

$$|2I_1| = \sqrt{1 - \frac{1}{M^2}}.$$  

(one is such that $-2I_1 = \sqrt{1 - 1/M^2}$, the other one such that $-2I_1 = -\sqrt{1 - 1/M^2}$. This proves the existence of functions $\kappa(\tau)$, such that we have equality in (3.10), and the sharpness of the bound is thus established. This concludes the proof of Proposition 1.

The remaining case $M < \frac{5}{3}$ is more difficult. We shall, as the main tool, use the theorem:

**Theorem 2.**

Let $t$ be a fixed function in $[0, t^*]$. Then the following holds:

$$|2I_1| < \frac{1}{2}$$

where $V(K)$ is.

**Proof:**

For $K \leq e^{-2t^*}$ with equality only if $\nu = 0$.

Equality holds if

$$\lambda(\tau) = e^{-\nu}$$

and no other functions.

For the case of $M > \frac{5}{3}$ goes back to Yamada's theory of universal functions. See a
Theorem 2.
Let \( t \) be a fixed positive number, \( \lambda(\tau) \) be a real, piecewise continuous function in \([0,t]\) and
\[
|\lambda(\tau)| \leq e^{-\tau}. \tag{3.12}
\]
Next, let \( K \) be a real number in \([0,(1-e^{-2t})/2]\). If
\[
\int_0^t \lambda^2(\tau)d\tau = K, \tag{3.13}
\]
then the following sharp bound holds:
\[
|\int_0^t \lambda(\tau)d\tau| \leq V(K), \tag{3.14}
\]
where \( V(K) \) is defined as follows:
For \( K \leq e^{-2t}t \) we have
\[
V(K) = \sqrt{K \cdot t}, \tag{3.15}
\]
with equality only for the constant function
\[
\lambda(\tau) = \sqrt{K/t} \quad \text{for} \quad 0 \leq \tau \leq t. \tag{3.15'}
\]
For \( K \geq e^{-2t}t \) we have
\[
V(K) = e^{-\nu(\nu + 1)} - e^{-t}, \tag{3.16}
\]
where \( \nu \in [0,t] \) is given by
\[
K = e^{-2\nu(\nu + 1)} - \frac{1}{2}e^{-2t}. \tag{3.17}
\]
Equality holds for the function
\[
\lambda(\tau) = e^{-\nu} \quad \text{for} \quad 0 \leq \tau \leq \nu \quad \text{and} \quad e^{-\tau} \quad \text{for} \quad \nu \leq \tau \leq t \tag{3.17'}
\]
and no other function.

For the case \( t = \infty \) (where the first subcase is empty) this theorem goes back to Valiron and Landau [6], and was used by Landau in the theory of univalent functions. The case presented above was established in the paper [11], and was there used in the theory of bounded univalent functions. See also [12].
For our purpose we choose
\[ \lambda(\tau) = \cos \vartheta(\tau) e^{-\tau} \quad \text{and} \quad t = \log M, \quad \text{where} \quad M < 5/3. \tag{3.18} \]

We thus have \( e^{-t} = 1/M \).

In (3.9) we may without loss of generality assume that \( I_2 \geq 0 \) and at the same time \( I_1 \geq 0 \). This may be seen as follows: If not, replace \( f \) by \( f_0 \), then \( I_1 \) is replaced by \( e^{-i\varphi} I_1 \) and \( I_2 \) by \( e^{-2i\varphi} I_2 \). By a proper choice of \( \varphi \) both expressions are non-negative. With
\[ \kappa(\tau) = \cos \vartheta(\tau) + i \sin \vartheta(\tau) \]
we have
\[ I_1 = \int_0^{\log M} \cos \vartheta(\tau) e^{-\tau} d\tau, \tag{3.19} \]
and
\[ 2I_2 = 2 \int_0^{\log M} \cos 2\vartheta(\tau) e^{-2\tau} d\tau = 4 \int_0^{\log M} \cos^2 \vartheta(\tau) e^{-2\tau} d\tau - (1 - \frac{1}{M^2}). \tag{3.20} \]

Moreover
\[ \int_0^{\log M} \sin \vartheta(\tau) e^{-\tau} d\tau = 0, \quad \text{and} \quad \int_0^{\log M} \sin 2\vartheta(\tau) e^{-2\tau} d\tau = 0. \tag{3.21} \]

The relation (3.9) thus takes the form (when we write \( \vartheta \) instead of \( \vartheta(\tau) \)):
\[ \left( \int_0^{\log M} \cos \vartheta \cdot e^{-\tau} d\tau \right)^2 = \int_0^{\log M} \cos^2 \vartheta \cdot e^{-2\tau} d\tau - \frac{1}{4} \left( 1 - \frac{1}{M^2} \right). \tag{3.22} \]

Now let \( K \) be the value of the integral
\[ \int_0^{\log M} \cos \vartheta e^{-2\tau} d\tau. \]

Then \( V(K) \) is the sharp upper bound for the integral
\[ \int_0^{\log M} \cos \vartheta e^{-\tau} d\tau. \]

The question now is: Given an \( M \)-value < 5/3. Is it possible to have (3.9) satisfied and at the same time have the maximal value of the second integral, i.e. is it possible to have
\[ V(K)^2 = K - \frac{1}{4} \left( 1 - \frac{1}{M^2} \right)? \tag{3.23} \]

It turns out that it is convenient to deal with the second case \( K \geq \log M/M^2 \) first.

\[ [e^{-\nu} \nu + \log M] \]

or
\[ [e^{-\nu} \nu + \log M] \]

Simple computation of the lefthand expression

which is positive if \( \nu \geq 0 \), is strictly increasing, is positive if \( \nu \geq 0 \), has at most one zero, is positive if \( \nu \geq 0 \)

For \( \nu = 0 \):

\[ (1 - \frac{1}{M^2}) \]

which is < 0 for \( M > 1 \)

For \( \nu = \log M \):

Simple computations show that the expression (3.24') has

at least one zero on the interval \( (1, M) \)

The only remaining open question, is the question whether (3.22) also (3.21) is satisfied for all \( M \) such that (3.22) is satisfied. Simple computation of (3.22) and (3.21) then shows that we go back to Theorem 2.8.
\[ \log M/M^2 \text{ first. In this case (3.23) takes the form} \]

\[ [e^{-\nu}(\nu + 1) - \frac{1}{M}]^2 = e^{-2\nu}(\nu + \frac{1}{2}) - \frac{1}{4}(1 + \frac{1}{M^2}), \]  

\[ (3.24) \]

or

\[ [e^{-\nu}(\nu + 1) - \frac{1}{M}]^2 - e^{-2\nu}(\nu + \frac{1}{2}) + \frac{1}{4}(1 + \frac{1}{M^2}) = 0. \]  

\[ (3.24') \]

Simple computation shows that the derivative with respect to \( \nu \) of the lefthand expression in (3.24') is

\[ 2\nu e^{-\nu}(\frac{1}{M} - \nu e^{-\nu}), \]

which is positive for all \( \nu \in (0, \log M) \). Hence the lefthand side of (3.24') is strictly increasing in the interval \((0, \log M)\), and the equation (3.24') has at most one solution in this interval. The values of the lefthand side of (3.24') in the endpoints are:

For \( \nu = 0 \):

\[ (1 - \frac{1}{M})^2 - \frac{3}{4}(1 - \frac{1}{M^2}) = (1 - \frac{1}{M})(\frac{3}{4} - \frac{5}{4M}), \]

which is \(< 0\) for all \( M \in (1, 5/3) \).

For \( \nu = \log M \):

\[ \frac{1}{M^2}((\log M)^2 - \log M + \frac{M^2}{4} - \frac{1}{4}). \]  

\[ (3.25) \]

Simple computation shows that there is a unique \( M \)-value \( M_0 = 1.361... \), such that the expression (3.25) is negative for all \( M \in (1, M_0) \) and positive for all \( M \in (M_0, 5/3) \). Keeping in mind the monotonicity property established above we have: For any \( M \) in the interval \((M_0, 5/3)\) the equation (3.24') has exactly one root in the interval \((0, \log M)\). For any \( M \) in the interval \((1, M_0)\) the equation (3.24') has no such root.

The only remaining question, as far as the interval \((M_0, 5/3)\) is concerned, is the question about a possible \( \kappa(\tau) \), such that, in addition to (3.22) also (3.21) is satisfied. The real parts \( \cos \vartheta \) and \( \cos 2\vartheta \) are already such that (3.22) is satisfied. Any change from \( \vartheta \) to \(-\vartheta \) does not violate the validity of (3.22). In order to find a change such that (3.21) is satisfied, we go back to Theorem 2, second case. From (3.17') with

\[ \lambda(\tau) = \cos \vartheta(\tau)e^{-\tau} \]

we find

\[ \]
\[ \sin \vartheta(\tau) e^{-\tau} = \pm \sqrt{e^{-2\tau} - e^{-2\nu}} \quad \text{for} \ 0 \leq \tau \leq \nu \quad \text{and} \ 0 \quad \text{for} \ \nu \leq \tau \leq t \]

and

\[ \sin 2\vartheta(\tau)e^{-2\tau} = 2e^{-\nu}e^{-\tau} \sin \vartheta(\tau) \quad \text{for} \ 0 \leq \tau \leq \nu \quad \text{and} \ 0 \quad \text{for} \ \nu \leq \tau \leq t. \]

Hence

\[ \int_{0}^{t} \sin 2\vartheta(\tau)e^{-2\tau}d\tau = 2e^{-\nu} \int_{0}^{t} \sin \vartheta(\tau)e^{-\tau}d\tau. \quad (3.26) \]

Take a \( \nu_0 \in [0, \nu] \) and let \( \vartheta \) be positive in \([0, \nu_0]\) and negative in \((\nu_0, \nu]\). By a proper choice of \( \nu_0 \)-value the "\( \sin \vartheta \)-integral" is 0, and, by (3.26) also the "\( \sin 2\vartheta \)-integral". Hence (3.21) is satisfied without violation of (3.22). We have thus proved the following:

**Proposition 3.**

Let \( M_0 \) be the unique \( M \)-value in \((1, 5/3)\) for which (3.25) is 0. Then for \( 1.316... = M_0 < M < 5/3 \) the following sharp bound holds in \( S'_M \):

\[ |a_2| \leq 2[e^{-\nu}(\nu + 1) - \frac{1}{M}], \quad (3.27) \]

where \( \nu \in [0, \log M] \) is uniquely given by (3.24').

In the remaining case \( 1 < M < M_0 \) we shall use the first case in Theorem 2, and (3.23) takes the form

\[ (\sqrt{K \log M})^2 = K - \frac{1}{4}(1 - \frac{1}{M^2}), \quad (3.28) \]

and hence

\[ K = \frac{1 - 1/M^2}{4(1 - \log M)}. \quad (3.28') \]

From Theorem 2 we know that we have equality in (3.14) if and only if

\[ \lambda(\tau) = \cos \vartheta(\tau)e^{-\tau} = \sqrt{\frac{K}{\log M}}. \]

From this we get

\[ \sin 2\vartheta(\tau)e^{-2\tau} = 2\sqrt{\frac{K}{\log M}} \sin \vartheta(\tau)e^{-\tau}, \]

showing that in this case \( \sin 2\vartheta(\tau)e^{-2\tau} \) and \( \sin \vartheta(\tau)e^{-\tau} \) only differ by a constant factor. This was, in the previous case the crucial point in proving that the two equalities in (3.21) can be satisfied simultaneously. This will, essentially in the same way, also hold in the present case. This shows that \( 2\sqrt{K \log M} \) is the sharp upper bound for \( |a_2| \), and we have the following result:

**Proposition 4.**

Let \( M_0 \) be the unique \( M \)-value in \((1, 5/3)\) for which (3.25) is 0. Then for \( 1 < M < M_0 \)

We put the result in:

**Theorem 5.**

Let \( A_2(M) \) be the following sharp bound holds:

For \( M \geq 5/3 \)

\[ |a_2| \leq 2[e^{-\nu}(\nu + 1) - \frac{1}{M}], \]

we have

\[ \lambda(\tau) = \cos \vartheta(\tau)e^{-\tau} = \sqrt{\frac{K}{\log M}}, \]

where \( \nu \in [0, \log M] \) is uniquely given by (3.24').

For \( 1 \leq M \leq M_0 \)

\[ K = \frac{1 - 1/M^2}{4(1 - \log M)}. \]

From (3.28')

**Remarks.**

a) It is a straightforward function of \( M \) and \( 5/3 \). This justifies the cases above.

b) It is not direct to check whether these functions in \( S'_M \)

is presently in \( S'_M \) in the case of Mushtaq.

c) The condition \( 2\sqrt{K \log M} = 2I_2 \) is not satisfied in this case, and a similar one in the present case.

This shows that \( 2\sqrt{K \log M} \) is the sharp upper bound for \( |a_2| \), and we have the following result:
Proposition 4.

Let \( M_0 \) be the unique \( M \)-value in \((1, 5/3)\) for which \((3.25)\) is 0. Then for \( 1 < M < M_0 = 1.316 \ldots \) the following sharp bound holds in \( S'_M \):

\[
|a_2| \leq \sqrt{\frac{(1 - 1/M^2) \log M}{1 - \log M}}.
\]  

(3.29)

We put the results in the Propositions 1, 3 and 4 together in one theorem:

Theorem 5.

Let \( A_2(M) \) be the maximum value of \( |a_2| \) in \( S'_M \). Then the following holds:

For \( M \geq 5/3 \) we have

\[
A_2(M) = \sqrt{1 - \frac{1}{M^2}}.
\]

For \( M_0 \leq M \leq 5/3 \), where \( M_0 = 1.316 \ldots \) is uniquely given in \((1, 5/3)\) by

\[
(\log M_0)^2 - \log M_0 - \frac{1}{4} + \frac{M_0^2}{4} = 0,
\]

we have

\[
A_2(M) = 2|e^{-\nu}(\nu + 1) - \frac{1}{M}|,
\]

where \( \nu \in [0, \log M] \) is given by

\[
|e^{-\nu}(\nu + 1) - \frac{1}{M}|^2 = e^{-2\nu(\nu + \frac{1}{2}) - \frac{1}{4}(1 + \frac{1}{M^2})}.
\]

For \( 1 \leq M \leq M_0 \) we have

\[
A_2(M) = \sqrt{\frac{(1 - 1/M^2) \log M}{1 - \log M}}.
\]

Remarks.

a) It is a straightforward verification to prove that \( A_2(M) \) is a continuous function of \( M \) in the interval \([1, \infty)\), the critical points being 1, \( M_0 \) and 5/3. This justifies the use of the equality signs in distinguishing the cases above.

b) It is not difficult to carry out the investigations above for \( k \)-symmetric functions in \( S'_M \) with vanishing third coefficient. Such an investigation is presently in progress by the Master of Science- student Muhammad Mushtaq.

c) The condition \( a_3 = 0 \) is equivalent to the condition \((3.9)\), i. e. to \( 4I_3^2 = 2I_2 \). Here the number 4 could be replaced by a positive number \( q \), and a similar investigation as the one in the present paper could be carried out, only with an additional discussion of which \( q \)-values are permitted.
REFERENCES.

Det Kongelige Norske Videnskabers Selskab ble stiftet i 1760 og Skrifter utkom første gang i 1761. Det er en av verdens eldste vitenskapelige skriftserier som fremdeles utkommer.

The Royal Norwegian Society of Sciences and Letters was founded in 1760 and the Transactions (Skrifter) first appeared in 1761. The Transactions series is among the oldest scientific publications in the world.