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# Sampling on Quasicrystals

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# Problem Description

The phenomenon of universal sets of sampling is studied. In particular, the article "Quasicrystals are sets of stable sampling" by Matei and Meyer (2008) is studied in detail, and the proofs of the text are worked through. Required background material is also discussed.

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## Preface

This paper constitutes my master's thesis, written at the Norwegian University of Science and Technology (NTNU). The thesis was written at the Department of Mathematical Sciences under the supervision of Professor Kristian Seip. Our agreement was that I would take a look at stable sampling and interpolation sets for certain function spaces. Ultimately I would work my way through the article *Quasicrystals are sets of stable sampling* by Matei and Meyer [9].

The thesis appears to me as a study of the work of three mathematicians: Beurling, Landau and Meyer. The proof presented in Matei and Meyer's article relies heavily on an early result by Beurling, so I started out studying Beurling's work. Encouraged by Seip I then did a thorough read-through of Landau's article *Necessary density conditions for sampling and interpolation of certain entire functions* [7]. The generality of the arguments presented by Landau astounds me, and I would highly recommend his article to others. It is simply a fun read. Finally, I studied the proof of the claim raised by Matei and Meyer, namely that quasicrystals are sets of stable sampling.

I am reasonably satisfied with the final result. Although it is *my* thesis, I like to think of it as a team effort, and a few thanks are in order. First, I thank my supervisor Kristian Seip for helpful discussions, and for bearing with me when returning with the same questions over and over again. A big thanks to my two proofreaders Jørgen Avdal and Emily Fertig, and to Hege Thalberg and Anders Nesbakken for keeping me company during late nights in the office. Lastly, I thank my boyfriend Henrik Enoksen for endless support and patience.

Trondheim, February 28, 2011.





**Abstract**

Let  $K \subset \mathbb{R}^n$  be compact and let  $B(K)$  be the set of square integrable functions whose Fourier transforms are supported on  $K$ . Let  $\Lambda \subset \mathbb{R}^n$  be a simple quasicrystal. If the density of the quasicrystal exceeds the Lebesgue measure of the set  $K$ , then  $\Lambda$  is a set of stable sampling for  $B(K)$ . A proof of this claim is provided. Necessary and sufficient density conditions for stable sampling and interpolation sets in one dimension are studied in detail.



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>2</b>
2.1	The Bessel inequality for the set $\mathcal{E}(\Lambda)$ . . . . .	3
2.2	Density of a uniformly discrete set . . . . .	4
2.3	Connecting density to stable sampling and interpolation . . . . .	5
2.4	Sampling on quasicrystals . . . . .	6
<b>3</b>	<b>Stable sampling and interpolation in one dimension</b>	<b>6</b>
3.1	The Paley-Wiener Theorem . . . . .	7
3.2	Stable sampling in one dimension . . . . .	7
3.2.1	Proof of Theorem 2.3 (B1) . . . . .	10
3.3	Stable interpolation in one dimension . . . . .	11
3.3.1	Proof of Theorem 2.3 (B2) . . . . .	12
<b>4</b>	<b>Landau's necessary density conditions</b>	<b>15</b>
4.1	An eigenvalue problem . . . . .	15
4.2	Connection to sets of stable sampling and interpolation . . . . .	18
4.3	Proofs of Theorems 4.1 and 4.2 . . . . .	21
<b>5</b>	<b>Density of Lattice Projections</b>	<b>23</b>
5.1	Definitions . . . . .	24
5.2	Ergodicity . . . . .	25
5.3	Determining the limit $d_K$ . . . . .	27
<b>6</b>	<b>Proof of Theorem 2.6</b>	<b>29</b>
<b>7</b>	<b>Further work</b>	<b>34</b>
<b>A</b>	<b>Existence of <math>D^-(\Lambda)</math> and <math>D^+(\Lambda)</math></b>	<b>37</b>
<b>B</b>	<b>Proof of the inclusion <math>B(I) \subseteq B(I)_0^\infty</math></b>	<b>38</b>
<b>C</b>	<b>Detailed calculation for Lemma 4.3 (v)</b>	<b>39</b>



## 1 Introduction

In the field of signal processing, a seemingly everlasting goal is finding cheap sampling protocols that allow for sparse sampling of signals without loss of information. The conventional approach to signal sampling follows the celebrated Shannon's sampling theorem; the sampling rate must be at least twice the maximum frequency present in the signal (known as the Nyquist rate). A few years back the theory of compressive sampling, or CS, was introduced by Candès, Romberg and Tao [2]. The CS theory asserts that certain signals can be sampled with no information loss at a much lower rate than indicated by Shannon's theorem. Furthermore, stable reconstruction from this sparse sampling is possible in feasible time. In order for compressive sampling to apply, the signal of interest must have a sparse representation in some convenient basis.

In the aftermath of the development of the CS theory, a few questions have naturally arisen; Are there perhaps other restrictions to put on a signal  $f$  that allow for extraordinarily sparse sampling without loss of information? Or can we obtain a cheap sampling protocol by being particularly clever when choosing our sampling set? Among those to be occupied with these questions were Matei and Meyer, who recently published the article *Quasicrystals are sets of stable sampling* [9]. Here they prove that sampling on so-called quasicrystals in  $\mathbb{R}^n$  ensures that no information is lost if the sampled signal has a compactly supported Fourier transform. Additionally, recovery of the signal by  $L^1$ -norm minimization is possible in special cases. The latter result is presented in the follow-up paper *A variant of compressed sensing* [10].

Let us explain more specifically what is claimed by Matei and Meyer in their first article. Let  $B(K)$  denote the space of square integrable functions whose Fourier transforms are supported on the compact set  $K \subset \mathbb{R}^n$ . If all functions in  $B(K)$  can be stably reconstructed from their sampling on a set  $\Lambda \subset \mathbb{R}^n$ , then  $\Lambda$  is said to be a *set of stable sampling* for  $B(K)$ . Matei and Meyer claim that a quasicrystal, if it satisfies certain density conditions, is a set of stable sampling for all function spaces  $B(K)$  where  $K$  is compact. This claim, and the proof of it, will be the main focus of this paper.

Matei and Meyer are not the first to study stable sampling sets for  $B(K)$ . It was Beurling [3] who first established that density plays an important role in determining whether or not a set can be a set of stable sampling for  $B(K)$ . Beurling was mainly concerned with one-dimensional sampling sets. In multiple dimensions, Landau's work on necessary density conditions for stable sampling sets stands out [7]. Matei and Meyer's article can be seen as a complement to Landau's work, as it points out a specific case in which Landau's necessary density conditions are in fact sufficient. We will soon elaborate further on the significance of Meyer and Matei's findings. First we give a precise definition of a stable sampling set. We also introduce the related term *stable interpolation set*, and present some preliminary density results by Beurling and Landau.

## 2 Preliminaries

Let  $K \subset \mathbb{R}^n$  be a compact set and  $B(K) \subset L^2(\mathbb{R}^n)$  be the translation invariant subspace of  $L^2(\mathbb{R}^n)$  consisting of all  $f \in L^2(\mathbb{R}^n)$  whose Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx \quad (2.1)$$

is supported on  $K$ . Let  $\Lambda$  be a subset of  $\mathbb{R}^n$ . We say that  $\Lambda$  is uniformly discrete if the minimum distance between any two distinct elements of  $\Lambda$ ,

$$\beta(\Lambda) = \inf_{\lambda, \gamma \in \Lambda, \lambda \neq \gamma} |\lambda - \gamma|, \quad (2.2)$$

exceeds some positive quantity. We term  $\beta(\Lambda)$  the separation of  $\Lambda$ .

**Definition 2.1.** *A uniformly discrete set  $\Lambda \subset \mathbb{R}^n$  has the property of stable sampling for  $B(K)$  if there exists a constant  $M < \infty$  such that*

$$f \in B(K) \quad \Rightarrow \quad \|f\|_2^2 \leq M \sum_{\lambda \in \Lambda} |f(\lambda)|^2. \quad (2.3)$$

We give an equivalent definition. Let  $L^2(K)$  be the space of all restrictions to  $K$  of functions in  $L^2(\mathbb{R}^n)$ . Then  $\Lambda \subset \mathbb{R}^n$  is a set of stable sampling for  $B(K)$  if and only if the collection of functions

$$\mathcal{E}(\Lambda) = \left\{ e^{2\pi i \lambda \cdot x} \mid \lambda \in \Lambda \right\} \quad (2.4)$$

is a frame of  $L^2(K)$ . That is, the frame condition

$$A \|f\|^2 \leq \sum_{\lambda \in \Lambda} \left| \langle f(x), e^{2\pi i \lambda \cdot x} \rangle \right|^2 \leq B \|f\|^2 \quad (2.5)$$

holds for all  $f \in L^2(K)$  with constants  $A, B < \infty$ . The right inequality above follows from the separation condition (2.2) on  $\Lambda$ , and will be justified in the following subsection.

**Definition 2.2.** *A set  $\Lambda \subset \mathbb{R}^n$  has the property of stable interpolation for  $B(K)$  if there exist constants  $A$  and  $B$  such that*

$$A \|f\|_{L^2(K)}^2 \leq \sum_{\lambda \in \Lambda} |c(\lambda)|^2 \leq B \|f\|_{L^2(K)}^2 \quad (2.6)$$

for every trigonometric sum  $f(x) = \sum_{\lambda \in \Lambda} c(\lambda) e^{2\pi i \lambda \cdot x}$  where  $(c(\lambda))_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ .

Just as the right inequality in (2.5), the left inequality in the above definition is due to the separation of  $\Lambda$ .

## 2.1 The Bessel inequality for the set $\mathcal{E}(\Lambda)$

As just mentioned, there are two ways to characterize a set of stable sampling. One can refer to Definition 2.1, or one may say that  $\Lambda$  is a set of stable sampling for  $B(K)$  if the set of functions  $\mathcal{E}(\Lambda)$  defined in (2.4) is a frame of  $L^2(K)$ . That the right inequality in (2.5) holds is simply a consequence of the set  $\Lambda$  being uniformly discrete, and is not related to the stable sampling definition. We give a short proof of this.

Note first that  $f \in L^2(K)$  if and only if  $\hat{f} \in B(K)$ . Furthermore, the sum in (2.5) is exactly the sum  $\sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2$ . By employing that  $\|f\|_2^2 = \|\hat{f}\|_2^2$ , we see that the right inequality in (2.5) holds for all functions  $f \in L^2(K)$  if and only if

$$\sum_{\lambda \in \Lambda} |g(\lambda)|^2 \leq A \|g\|_2^2 \quad (2.7)$$

holds for all functions  $g \in B(K)$ .

Let  $\beta$  denote the separation of the set  $\Lambda$  as given by (2.2). We define the function

$$h(x) = C \prod_{j=1}^n h_0(x_j), \quad (2.8)$$

where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $h_0(x_j) = I_{[-\epsilon, \epsilon]}(x_j)$  and  $I_X$  denotes the characteristic function of the set  $X \subset \mathbb{R}$ .  $\epsilon$  is fixed sufficiently small for  $h(x)$  to be supported on  $|x| < \beta/2$ , and we adjust the constant  $C$  such that the Fourier transform  $\hat{h}$  satisfies  $|\hat{h}(x)| \geq 1$  for all  $x \in K$ . Given any function  $g \in B(K)$ , we can construct a function  $f \in B(K)$  such that

$$g(x) = \int_{\mathbb{R}^n} f(y) h(x-y) dy = \int_{|x-y| < \beta/2} f(y) h(x-y) dy.$$

From Parseval's Theorem and the properties of the function  $h$  it follows that

$$\|g\|_2^2 = \|\hat{g}\|_2^2 = \|\hat{f} \cdot \hat{h}\|_2^2 \geq \|\hat{f}\|_2^2 = \|f\|_2^2,$$

and by the Cauchy-Schwarz inequality we get

$$|g(x)|^2 \leq \|h\|_2^2 \int_{|x-y| < \beta/2} |f(y)|^2 dy.$$

Combining the above inequalities we arrive at (2.7), namely

$$\sum_{\lambda \in \Lambda} |g(\lambda)|^2 \leq \|h\|_2^2 \sum_{\lambda \in \Lambda} \left( \int_{|\lambda-y| < \beta/2} |f(y)|^2 dy \right) \leq \|h\|_2^2 \|f\|_2^2 \leq A \|g\|_2^2,$$

with  $A = \|h\|_2^2$ . We will always assume that  $\Lambda$  is uniformly discrete, and thus may always make use of inequality (2.7) for functions in  $B(K)$ .

With the Bessel inequality established for the set  $\mathcal{E}(\Lambda)$ , we proceed to verify the left inequality in (2.6). Choose any sequence  $(c(\lambda))_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ . For  $f(x) = \sum_{\lambda \in \Lambda} c(\lambda) e^{2\pi i \lambda \cdot x}$  we have

$$\begin{aligned} \|f\|_2^2 &= \left| \sum_{\lambda \in \Lambda} c(\lambda) \langle f, e^{2\pi i \lambda \cdot x} \rangle \right| \\ &\leq \left( \sum_{\lambda \in \Lambda} |c(\lambda)|^2 \right)^{\frac{1}{2}} \left( \sum_{\lambda \in \Lambda} |\langle f, e^{2\pi i \lambda \cdot x} \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq B \|f\|_2 \left( \sum_{\lambda \in \Lambda} |c(\lambda)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where Bessel's inequality for  $\mathcal{E}(\Lambda)$  is used in the last inequality. Dividing both sides by  $B \|f\|_2$  and squaring terms we arrive at

$$C \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |c(\lambda)|^2,$$

where  $C < \infty$ . Thus, the left inequality in (2.6) holds for any uniformly discrete set  $\Lambda$ .

## 2.2 Density of a uniformly discrete set

When defining the density of a uniformly discrete set, we consider first a one-dimensional set  $\Lambda$  in  $\mathbb{R}$ . The elements of  $\Lambda$  are numbered such that  $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$  and  $\lambda_j < \lambda_{j+1}$  for all  $j \in \mathbb{Z}$ . Let  $n^-(r)$  and  $n^+(r)$  denote respectively the smallest and largest number of elements of  $\Lambda$  to be found in an interval of length  $r$ . We define the lower and upper uniform densities of  $\Lambda$  by

$$D^-(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^-(r)}{r} \quad \text{and} \quad D^+(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^+(r)}{r}. \quad (2.9)$$

The superadditivity of  $n^-(r)$  and subadditivity of  $n^+(r)$  ensure that the limits exist. A proof of this claim is found in Appendix A. If  $D^-(\Lambda)$  and  $D^+(\Lambda)$  are equal, we put  $D^-(\Lambda) = D^+(\Lambda) = D(\Lambda)$ , and refer to  $D(\Lambda)$  as the density of  $\Lambda$ .

For higher-dimensional, uniformly discrete sets  $\Lambda \subset \mathbb{R}^n$  the density definition is more intricate. When measuring density in one dimension, we do so by determining the largest and smallest number of elements  $n^\pm(r)$  of  $\Lambda$  to be found in a translate of  $rI$ , where  $I$  is the unit interval. In more than one dimension, the choice of  $I$  represents an additional element of freedom: for each set  $I \subset \mathbb{R}^n$  of measure 1 we may define  $n^\pm(r) = n^\pm(rI)$  to be the largest and smallest number of elements of  $\Lambda$  to be found in a translate of  $rI$ . Following the definition of density in one dimension we would then let

$$D^-(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^-(rI)}{r^n} \quad \text{and} \quad D^+(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^+(rI)}{r^n}. \quad (2.10)$$

How do we know which  $I$  is the appropriate choice in the definitions above? The answer to this question is that any Riemann measurable, compact set  $I$  of measure 1 will do [7,



Lemma 4]. The set  $I$  is Riemann measurable if the measure of its boundary is zero. The definitions  $D^\pm(\Lambda)$  in (2.10) are independent of  $I$  under these mild regularity conditions, and accordingly we can speak unambiguously of the upper and lower uniform densities of  $\Lambda$  using (2.10). Whenever the upper and lower uniform densities coincide we let  $D^-(\Lambda) = D^+(\Lambda) = D(\Lambda)$ , and refer to  $D(\Lambda)$  as the density of  $\Lambda$ .

### 2.3 Connecting density to stable sampling and interpolation

Consider the space  $B(I)$  of functions whose Fourier transforms are supported on the one-dimensional interval  $I = (-a, a) \subset \mathbb{R}$ . We denote by  $|\cdot|$  the Lebesgue measure of a set. The following theorem was stated by Beurling.

**Theorem 2.3.** *Let  $\Lambda \subset \mathbb{R}$  be a uniformly discrete, increasing sequence of real numbers  $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ . Then*

(B1)  $D^-(\Lambda) > |I|$  implies that  $\Lambda$  is a set of stable sampling for  $B(I)$ , and

(B2)  $D^+(\Lambda) < |I|$  implies that  $\Lambda$  is a set of stable interpolation for  $B(I)$ .

This theorem justifies our recent discussion of density. We see that the density of  $\Lambda$  alone determines whether it is a set of stable sampling or interpolation for the function space  $B(I)$ . A detailed proof of Theorem 2.3 is given in Section 3.

If the inequalities in (B1) and (B2) were not strict, then Theorem 2.3 would have stated both sufficient *and* necessary density bounds on the set  $\Lambda$ . A stable sampling set  $\Lambda$  will indeed satisfy  $D^-(\Lambda) \geq |I|$ . Likewise we know that  $D^+(\Lambda) \leq |I|$  for any stable interpolation set for  $B(I)$ . When moving to higher dimensions, sufficient density conditions for  $\Lambda$  are harder to find. The following theorem on necessary density conditions is due to Landau [7].

**Theorem 2.4.** *Let  $K \subset \mathbb{R}^n$  be any compact set, and let  $\Lambda \subset \mathbb{R}^n$  be a uniformly discrete set.*

(L1) *If  $\Lambda$  is a set of stable sampling for  $B(K)$ , then  $D^-(\Lambda) \geq |K|$ .*

(L2) *If  $\Lambda$  is a set of stable interpolation for  $B(K)$ , then  $D^+(\Lambda) \leq |K|$ .*

We give a detailed proof of Theorem 2.4 in Section 4.

(L1) and (L2) are not if and only if statements. Indeed  $|K| \leq D^-(\Lambda)$  does not even imply (2.3) when  $\Lambda = \mathbb{Z}^n$ . However, the question naturally arises of whether there are restrictions to put on  $\Lambda$  which would render the converses of (L1) and (L2) true. Do there exist sets  $\Lambda$  which are stable sampling sets for *all* spaces  $B(K)$  where the measure of  $K$  does not exceed  $D^-(\Lambda)$ ?

**Definition 2.5.** *We say that the uniformly discrete set  $\Lambda$  is a universal set of stable sampling if  $\Lambda$  has uniform density  $D(\Lambda)$  and is a set of stable sampling for  $B(K)$  for every compact set  $K$  of Lebesgue measure less than  $D(\Lambda)$ .*

Likewise, we say that  $\Lambda$  is a universal set of stable interpolation if  $\Lambda$  has uniform density  $D(\Lambda)$  and is a set of stable interpolation for every  $B(K)$  where the compact set  $K$  has Lebesgue measure exceeding  $D(\Lambda)$ .

The term *universal sampling set* was first introduced by Olevskii and Ulanovskii [11], who proved that universal sampling sets do not exist if  $K$  is allowed to be an arbitrary Borel set of measure less than the density of the sampling set. Accordingly, the restriction that  $K$  be compact is included in the definitions above.

## 2.4 Sampling on quasicrystals

The question was just raised of what restrictions must be put on a set  $\Lambda$  in order for it to be a universal set of stable sampling or interpolation. This brings us to the main theorem of this paper.

**Theorem 2.6.** [9, Theorem 1.4] *Let  $\Lambda \subset \mathbb{R}^n$  be a simple quasicrystal and  $K \subset \mathbb{R}^n$  be any compact set. If  $|K| < D(\Lambda)$ , then  $\Lambda$  is a set of stable sampling for  $B(K)$ . Furthermore, if  $K$  is Riemann integrable and  $|K| > D(\Lambda)$ , then  $\Lambda$  is a set of stable interpolation for  $B(K)$ .*

A quasicrystal is defined as follows [9]. Let  $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$  be a lattice. For an element  $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ , denote the projections onto  $\mathbb{R}^n$  and  $\mathbb{R}$  by  $p_1(x, t) = x$  and  $p_2(x, t) = t$ . We assume that  $p_1$  when restricted to  $\Gamma$  is an injective mapping onto the dense subset  $p_1(\Gamma)$  of  $\mathbb{R}^n$ . Likewise, the mapping  $p_2$  restricted to  $\Gamma$  is assumed injective onto  $p_2(\Gamma)$ , with  $p_2(\Gamma)$  dense in  $\mathbb{R}$ . If  $I = [-\alpha, \alpha]$ , then the simple quasicrystal  $\Lambda_\alpha \subset \mathbb{R}^n$  is given by

$$\Lambda_\alpha = \{p_1(\gamma) \mid \gamma \in \Gamma, p_2(\gamma) \in I\} . \quad (2.11)$$

Theorem 2.6 is due to Matei and Meyer [9], and claims the quite astonishing fact that quasicrystals are universal sets of stable sampling. Any function  $f \in B(K)$  can be stably reconstructed from its sampling on a quasicrystal, so long as  $K$  is a compact set with Lebesgue measure not exceeding the density of the quasicrystal. The proof of Theorem 2.6 is presented in Section 6. Prior to this we discuss density and equidistribution properties of lattice projections onto lower-dimensional subspaces in Section 5. The arguments and results given here will play an essential role when we finally prove Theorem 2.6.

## 3 Stable sampling and interpolation in one dimension

We turn to the special case in one dimension where  $\Lambda = (\lambda_j)_{j \in \mathbb{Z}} \subset \mathbb{R}$  is a uniformly discrete set and  $K$  is the interval  $I = (-a, a)$ . Our aim is to give a proof of Theorem 2.3. The two statements of this theorem will be treated separately, and we start by considering (B1) concerning stable sampling. First, however, we make a general observation on the one-dimensional function space  $B(I)$ .

### 3.1 The Paley-Wiener Theorem

The function space  $B(I)$  is the collection of all square integrable functions whose Fourier transforms are supported on the interval  $I = (-a, a)$ . Let  $PW_{2\pi a}$  denote the Paley-Wiener space of all entire functions of exponential type at most  $2\pi a$  which are square integrable on the real line. The following theorem, known as the Paley-Wiener Theorem, claims that  $B(I)$  and  $PW_{2\pi a}$  are in fact the same function space.

**Theorem 3.1.** [4, p. 158] *A function  $f$  belongs to  $PW_{2\pi a}$  if and only if it can be represented as*

$$f(z) = \int_{-a}^a \hat{f}(\xi) e^{2\pi i \xi z} d\xi$$

for some  $\hat{f} \in L^2(-a, a)$ .

Theorem 3.1 allows us to choose whether we want to prove Theorem 2.3 for functions of  $B(I)$  or functions of  $PW_{2\pi a}$ . We note that  $PW_{2\pi a}$  is a reproducing kernel Hilbert space when endowed with the usual inner product on  $L^2(\mathbb{R})$ . The reproducing kernel of  $PW_{2\pi a}$  is

$$k_x(y) = k(x, y) = 2a \operatorname{sinc}(2a(y - x)) ,$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x} .$$

This means that

$$f(x) = \langle f, k_x \rangle$$

for any  $f \in PW_{2\pi a}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product on  $L^2(\mathbb{R})$ .

We make the additional observation that  $B(K)$  is a reproducing kernel Hilbert space for any compact  $K \subset \mathbb{R}$ . The reproducing kernel of  $B(K)$  is

$$k_x(y) = \int_K e^{2\pi i(y-x)t} dt . \quad (3.1)$$

### 3.2 Stable sampling in one dimension

The first statement of Theorem 2.3 is proven in two steps. Ultimately, we deal with the function space  $B(I)$ , and show that (B1) holds. Prior to this we treat the larger space of continuous, bounded functions  $f(x)$  which tend to zero as  $|x| \rightarrow \infty$  and whose Fourier transforms are supported on  $I = (-a, a)$ . We denote this space by  $B(I)_0^\infty$ . A proof that  $B(I) \subseteq B(I)_0^\infty$  is found in Appendix B.

We define  $M(I, \Lambda)$  as the smallest  $M$  such that

$$\|f\|_\infty \leq M \|f|_\Lambda\|_\infty = M \sup_{\lambda_j \in \Lambda} |f(\lambda_j)| \quad (3.2)$$

for all  $f \in B(I)_0^\infty$  and a fixed, uniformly discrete set  $\Lambda \subset \mathbb{R}$ . This  $M$  might be infinite. We claim, however, that  $M$  is finite under certain restrictions on the lower uniform density of  $\Lambda$ .

**Theorem 3.2.** [3, p. 346, Theorem 5] *If  $D^-(\Lambda) > 2a$ , then  $M(I, \Lambda) < \infty$ .*

We make a few observations on  $B(I)_0^\infty$ . Any function  $f \in B(I)_0^\infty$  extends to an entire function of exponential type  $2\pi a$  on  $\mathbb{C}$ , and by Bernstein's inequality we have that [8, p. 227]

$$|f'(x_0)| \leq 2\pi a \sup_{\mathbb{R}} |f(x)|, \quad x_0 \in \mathbb{R}. \quad (3.3)$$

Furthermore, if  $(f_n)$  is a sequence of functions contained in the ball

$$\left\{ f \in B(I)_0^\infty \left| \sup_{\mathbb{R}} |f(x)| \leq M \right. \right\},$$

then there exists a subsequence  $(f_{n_k})$  converging pointwise and uniformly on compact sets to some function  $f$  in this ball. This follows from a normal family argument [1, p. 224] combined with the Phragmén-Lindelöf principle, and will be referred to as the compactness property of  $B(I)_0^\infty$ .

We introduce Beurling's notion of weak limits of the set  $\Lambda$  [3, p. 343]. Let  $(x_n)$  be an arbitrary sequence of real numbers and define an associated sequence of translates of  $\Lambda$  by

$$\Lambda_n = \Lambda + x_n.$$

We say that  $\Lambda_n$  converges weakly to  $\Lambda' = (\lambda'_j)$  if we can index  $\Lambda_n = (\lambda_j^n)$  in such a way that  $\lambda_j^n \rightarrow \lambda'_j$  as  $n \rightarrow \infty$  for every  $j$  in the index set of  $\Lambda'$ , and  $|\lambda_j^n| \rightarrow \infty$  as  $n \rightarrow \infty$  for those  $j$  which are not in this index set. When  $\Lambda$  is uniformly discrete, then every sequence of translates  $\Lambda_n$  contains a subsequence  $\Lambda_{n_k}$  converging weakly to some uniformly discrete set  $\Lambda'$  (which may be finite or even empty). This follows from splitting  $\mathbb{R}$  into intervals containing at most one point of each  $\Lambda_n$ , and using the fact that closed intervals are compact. We denote weak convergence by  $\Lambda_n \rightarrow \Lambda'$ , and let  $W(\Lambda)$  be the collection of all weak limits of  $\Lambda$ . The set  $W(\Lambda)$  possesses the following property.

**Lemma 3.3.** [14, Lemma 3.13] *Fix an interval  $I = (-a, a)$  and let  $\Lambda \subset \mathbb{R}$  be uniformly discrete. Then*

$$M(\Lambda') \leq M(\Lambda).$$

for every  $\Lambda' \in W(\Lambda)$ , where  $M(\Lambda) = M(I, \Lambda)$  as defined in (3.2).

*Proof.* Fix some  $\Lambda' \in W(\Lambda)$  and say  $\Lambda_n = \Lambda + x_n \rightarrow \Lambda'$ . It follows from the translation invariance of  $B(I)_0^\infty$  that  $M(\Lambda_n) = M(\Lambda)$  for every  $n$ . Choose  $f \in B(I)_0^\infty$  such that

$$\|f\|_\infty = 1 \quad \text{and} \quad \|f|_{\Lambda'}\|_\infty \leq M^{-1} + \epsilon,$$

where  $M = M(\Lambda')$  is possibly infinite. Let  $\lambda_j^n \in \Lambda_n \cap I_N$ , where  $I_N = [-N, N]$ . Then from Bernstein's inequality (3.3) and the fact that  $\Lambda_n \rightarrow \Lambda'$  we get

$$|f(\lambda_j^n)| \leq |f(\lambda'_j)| + 2\pi a |\lambda_j^n - \lambda'_j| < |f(\lambda'_j)| + \epsilon$$

for all sufficiently large  $n$ . We choose  $N$  such that  $|f(\lambda_j^n)| < \epsilon$  for  $\lambda_j^n \in I_N^C$ . Hence, we have that  $\|f|_{\Lambda_n}\|_\infty < \|f|_{\Lambda'}\|_\infty + \epsilon$  for all  $n$  larger than some threshold, or equivalently that  $M(\Lambda) = M(\Lambda_n) \geq (M^{-1} + 2\epsilon)^{-1}$ . Since  $\epsilon > 0$  was arbitrary, the lemma follows.  $\square$

**Lemma 3.4.** [3, p. 345, Theorem 3] Let  $M(I, \Lambda)$  be as given by (3.2). We have that  $M(I, \Lambda) < \infty$  if and only if

$$\Lambda_0 \in W(\Lambda) , f \in B(I)_0^\infty \text{ and } f|_{\Lambda_0} = 0 \quad \Rightarrow \quad f \equiv 0 .$$

*Proof.* Assume  $M(I, \Lambda) < \infty$ . Then by Lemma 3.3 we have  $M(\Lambda_0) \leq M(\Lambda)$ , and any  $f \in B(I)_0^\infty$  satisfying  $f = 0$  on  $\Lambda_0$  must necessarily be the zero function  $f \equiv 0$ .

Conversely, assume  $M(I, \Lambda) = \infty$ . Then there exists a sequence  $(f_n) \in B(I)_0^\infty$  such that  $\sup_{\mathbb{R}} |f_n(x)| = 1$  and  $\sup_{\Lambda} |f_n(x)| \rightarrow 0$ . Choose  $x_n$  such that  $|f_n(x_n)| = 1/2$ , and define  $g_n(x) = f_n(x - x_n)$ . Then  $|g_n(0)| = 1/2$ . Setting  $\Lambda_n = \Lambda + x_n$ , we have that

$$\sup_{\Lambda_n} |g_n(x)| \rightarrow 0 .$$

Let  $\Lambda_0$  be the weak limit of  $\Lambda_n$  (which might be empty). By the compactness property we may assume that  $g_n$  converges pointwise to some  $g \in B(I)_0^\infty$ . Let  $x_j^0$  be an element of  $\Lambda_0$ , and say  $x_j^n \rightarrow x_j^0$  as  $n \rightarrow \infty$  where  $x_j^n \in \Lambda_n$ . We have

$$|g_n(x_j^0)| \leq |g_n(x_j^0) - g_n(x_j^n)| + |g_n(x_j^n)| \rightarrow 0 ,$$

where the first term on the right hand side tends to zero by Bernstein's inequality. It follows that  $g = 0$  on  $\Lambda_0$ , but clearly  $|g(0)| = 1/2$ . This completes the proof of the converse claim.  $\square$

With Lemma 3.4 established we are prepared to prove Theorem 3.2. Assume that  $D^-(\Lambda) > 2a$ . Consider  $\Lambda_0 \in W(\Lambda)$ , and fix a function  $f \in B(I)_0^\infty$  which is zero when restricted to  $\Lambda_0$ . We wish to show that  $f \equiv 0$ . Aiming at a proof by contradiction, we assume that  $f$  is not identically zero. Without loss of generality we can then assume  $f(0) \neq 0$  and  $\sup_{\mathbb{R}} |f(x)| = 1$ . Recall that the function  $f$  extends to an entire function of exponential type  $2\pi a$  on  $\mathbb{C}$ . Let  $N_0(r)$  be the number of points of  $\Lambda_0$  in the interval  $[-r, r]$ . By Jensen's formula [1, p. 204] we have

$$\int_0^r \frac{N_0(t)}{t} dt \leq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|f(re^{i\theta})|}{|f(0)|} d\theta \leq 4ar - \log |f(0)| .$$

Furthermore, it is clear that  $n^-(2r) \leq N_0(r)$ , so

$$\int_0^{2r} \frac{n^-(t)}{t} dt = \int_0^r \frac{n^-(2t)}{t} dt \leq \int_0^r \frac{N_0(t)}{t} dt .$$

By taking derivatives we get

$$\lim_{r \rightarrow \infty} \frac{n^-(r)}{r} \leq 2a .$$

The left hand side in the above equation is the lower uniform density  $D^-(\Lambda)$  as defined in (2.9). We have reached a contradiction, so our assumption must be wrong. We conclude that  $f \equiv 0$ , and by Lemma 3.4 it follows that  $M(I, \Lambda) < \infty$ .  $\square$

### 3.2.1 Proof of Theorem 2.3 (B1)

Theorem 3.2 concerns functions in  $B(I)_0^\infty$ . We now return to the smaller space  $B(I)$  of square integrable functions with Fourier transforms supported on  $I = (-a, a)$ . Theorem 3.2 may be exploited to make an even stronger statement about functions in this space. We are ready to prove Theorem 2.3 (B1).

Denote by  $c_0$  the Banach space of sequences converging to zero, and let  $X$  be the closed subspace of  $c_0$  given by

$$X = \left\{ (f(\lambda_j))_{j \in \mathbb{Z}} \mid f \in B(I)_0^\infty \right\} .$$

Under the assumption  $D^-(\Lambda) > 2a$ , it follows from Theorem 3.2 that point evaluation of  $f \in B(I)_0^\infty$  at some fixed  $x \in \mathbb{R}$  is a bounded linear functional on  $X$ . We may write

$$f(x) = \sum_{\lambda_j \in \Lambda} c_j(x) f(\lambda_j) , \quad (3.4)$$

where  $(c_j(x)) \in \ell^1(\mathbb{Z})$  for each  $x \in \mathbb{R}$ .

If  $D^-(\Lambda) > 2a$ , then there exists some  $\epsilon > 0$  such that  $D^-(\Lambda) > 2a + 2\epsilon$ . It follows from Theorem 3.2 that the bounded linear functional (3.4) could just as well have been defined on  $\tilde{X}$ , where

$$\tilde{X} = \left\{ (f(\lambda_j))_{j \in \mathbb{Z}} \mid f \in B(\tilde{I})_0^\infty \right\} ,$$

and  $\tilde{I} = (-a - \epsilon, a + \epsilon)$ ; suppose it were. Choose any  $f \in B(I) \subset B(I)_0^\infty$ , and let

$$g(\xi) = \left( \frac{\sin \tilde{\epsilon}(x - \xi)}{\tilde{\epsilon}(x - \xi)} \right) f(\xi) .$$

For some sufficiently small  $\tilde{\epsilon} > 0$ , we have that  $g \in B(\tilde{I})_0^\infty$ . Applying the constructed functional (3.4) we have

$$g(x) = f(x) = \sum_j c_j(x) \frac{\sin \tilde{\epsilon}(x - \lambda_j)}{\tilde{\epsilon}(x - \lambda_j)} f(\lambda_j) ,$$

and using the Cauchy-Schwarz inequality we get

$$\|f\|_2^2 \leq \frac{\pi}{\tilde{\epsilon}} \cdot \sup_{x \in \mathbb{R}} \left( \sum_j |c_j(x)|^2 \right) \cdot \sum_j |f(\lambda_j)|^2 \leq C \sum_j |f(\lambda_j)|^2$$

where  $C$  is independent of  $f$ . As  $f \in B(I)$  was arbitrary, we have established that there exists a constant  $C < \infty$  such that

$$\|f\|_2^2 \leq C \sum_j |f(\lambda_j)|^2$$

for all  $f \in B(I)$  whenever  $D^-(\Lambda) > 2a$ . This completes the proof of Theorem 2.3 (B1).  $\square$

### 3.3 Stable interpolation in one dimension

We turn to the second claim of Theorem 2.3 concerning sets of stable interpolation. Matei and Meyer state that  $\Lambda \subset \mathbb{R}$  is a set of stable interpolation for  $B(I)$  if for all functions  $f(x) = \sum_j c_j e^{2\pi i \lambda_j x}$  where  $(c_j) \in \ell^2(\mathbb{Z})$  we have

$$A \|f\|_{L^2(I)}^2 \leq \sum_j |c_j|^2 \leq B \|f\|_{L^2(I)}^2$$

for constants  $A, B < \infty$  [9]. In other sources, this is referred to as the set  $\mathcal{E}(\Lambda)$  of exponentials defined in (2.4) being a Riesz sequence in  $L^2(I)$  [14]. A more common definition of  $\Lambda$  as a set of stable interpolation for  $B(I)$  is the following. Say  $w = (w_j)_{j \in \mathbb{Z}}$  is an arbitrary sequence in  $\ell^2(\mathbb{Z})$ . Then  $\Lambda$  is a set of stable interpolation for  $B(I)$  if for every such sequence  $w$  there exists a function  $f \in B(I)$  such that

$$f(\lambda_j) = w_j \quad \forall j \in \mathbb{Z}.$$

This alternative definition of a stable interpolation set does not apply to  $B(I)$  exclusively. We prove that the two definitions given above are in fact equivalent for any space  $B(K)$  where  $K \subset \mathbb{R}$  is compact.

**Proposition 3.5.** [15, p.23] *The following two statements are equivalent.*

- (1) *For every sequence  $(w_j) \in \ell^2(\mathbb{Z})$  there exists a function  $f \in B(K)$  such that  $f(\lambda_j) = w_j$  for all  $j \in \mathbb{Z}$ .*
- (2) *There exist constants  $A, B < \infty$  such that (2.6) holds for every sum  $f(x) = \sum_j c_j e^{2\pi i \lambda_j x}$  where  $(c_j) \in \ell^2(\mathbb{Z})$ .*

*Proof.* Note first that by Parseval's identity (2) is equivalent to stating that there exist constants  $A, B < \infty$  such that

$$A \left\| \sum_j c_j k_{\lambda_j} \right\|_2^2 \leq \sum_j |c_j|^2 \leq B \left\| \sum_j c_j k_{\lambda_j} \right\|_2^2, \quad (3.5)$$

where  $k_{\lambda_j}$  are the reproducing kernels (3.1) of  $B(K)$ . Identify the orthogonal complement in  $B(K)$  of the space of functions vanishing on  $\Lambda$  as the closure of the span of the set  $\{k_{\lambda_j} \mid j \in \mathbb{Z}\}$ . Denote this space by  $\mathcal{K}$ . Let  $T$  be the map from  $\ell^2(\mathbb{Z})$  to  $\mathcal{K}$  given by  $c_j \rightarrow \sum_j c_j k_{\lambda_j}$ , and denote by  $S$  the map  $f \rightarrow (f(\lambda_j))_{j \in \mathbb{Z}}$  between  $\mathcal{K}$  and  $\ell^2(\mathbb{Z})$ . Observe that  $S$  and  $T$  are Hilbert-adjoint operators. Accordingly,  $S$  has a bounded inverse if and only if  $T$  has a bounded inverse. By (3.5) it follows that  $T$  has a bounded inverse. Statement (1) implies that  $S$  has a bounded inverse by Banach's Open Mapping Theorem. Thus, (1) and (2) must be equivalent.  $\square$

### 3.3.1 Proof of Theorem 2.3 (B2)

In proving Theorem 2.3 (B2) we exploit Theorem 3.1, and refer to functions of  $PW_{2\pi a}$  rather than  $B(I)$ . Given a uniformly discrete set  $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$  for which  $D^+(\Lambda) < 2a$ , we simply construct the function  $f \in PW_{2\pi a}$  which satisfies  $f(\lambda_j) = w_j$  for a given sequence  $(w_j) \in \ell^2(\mathbb{Z})$ . We follow the argument given by Beurling [3, p. 351–365]. Let  $d = D^+(\Lambda) < 2a$ , and choose some rational  $d_1$  such that  $d < d_1 < 2a$ . From the definition (2.9) of upper uniform density it is clear that we may choose  $L$  such that  $n^+(L)/L < d_1$ . Then every interval of length  $L$  contains at most  $Ld_1$  points. Suppose without loss of generality that  $m = Ld_1$  is a natural number, and fix  $L$  accordingly. Let  $\{\omega_k \mid k \in \mathbb{Z}\}$  be a subdivision of  $\mathbb{R}$  into intervals of length  $L$ , where 0 is the common endpoint of  $\omega_0$  and  $\omega_1$ . If some interval  $\omega_k$  contains less than  $m$  points, add more points to this interval while ensuring that  $\Lambda$  combined with all new points remains uniformly discrete. The expanded set of points is still denoted  $\Lambda$ .

**Lemma 3.6.** [3, p.357, Lemma 7] *Let  $\Lambda$  be as above and assume  $0 \in \Lambda$ . The limit*

$$f(z) = \lim_{R \rightarrow \infty} \left\{ \prod_{0 < |\lambda| < R, \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) \right\}$$

*exists for all  $z \in \mathbb{C}$ . Additionally,  $f$  is an entire function vanishing on  $\Lambda \setminus \{0\}$ ,  $f(0) = 1$  and*

$$|f(x + iy)| \leq C(|z| + 1)^{4m} e^{\pi d_2 |y|}, \quad d_2 < 2a. \quad (3.6)$$

*Proof.* We start out by showing that the limit exists. Fix  $z$  and consider only  $|\lambda| > 2|z|$ . We get that

$$\sum_{|\lambda| < R} \log \left(1 - \frac{z}{\lambda}\right) = -z \sum_{|\lambda| < R} \lambda^{-1} + O\left(|z|^2 \sum_{|\lambda| < R} \frac{1}{|\lambda|^2}\right).$$

By pairing intervals  $\omega_k$  we have that

$$\left| \sum_{|\lambda| < R} \lambda^{-1} - \sum_{|\lambda| < R'} \lambda^{-1} \right| \leq \left( \sum_{k=s}^{s+N-1} \frac{m}{k(k-1)L} \right) + \frac{2m}{L(s-1)}, \quad (3.7)$$

where  $s$  is the index of the interval  $\omega_s$  containing  $R'$  and  $N$  is the number of intervals  $\omega_k$  completely contained in  $(R', R)$ . As  $R, R' \rightarrow \infty$ , we have that  $s \rightarrow \infty$  and the right hand side of (3.7) tends to zero. This proves that  $f(z)$  exists.

When proving the estimate (3.6), we treat each quadrant of  $\mathbb{C}$  separately. For a real  $x > 0$  we find  $k$  such that  $x \in \omega_k$ , and write  $|f(x)|$  as

$$|f(x)| = \prod_{\lambda \in \Lambda} \left|1 - \frac{x}{\lambda}\right| = \prod_{l=-\infty}^{\infty} \left( \prod_{\lambda \in \omega_l} \left|1 - \frac{x}{\lambda}\right| \right).$$



We bound the above product as follows. For  $l > k$ , all points  $\lambda \in \omega_l$  are placed at the right endpoint of  $\omega_l$ . Likewise, for  $1 < l < k$ , all points  $\lambda \in \omega_l$  are placed at the left endpoint of  $\omega_l$ . We disregard  $\omega_k$ ,  $\omega_1$  and  $\omega_0$ , and for  $l < 0$  we place all points  $\lambda \in \omega_l$  at the right endpoint of  $\omega_l$ . We then get

$$|f(x)| \leq \prod_{\omega_0} \left|1 - \frac{x}{\lambda}\right| \cdot \prod_{\omega_1} \left|1 - \frac{x}{\lambda}\right| \cdot \prod_{\omega_k} \left|1 - \frac{x}{\lambda}\right| \cdot \frac{\prod_{l=-\infty, l \neq 0}^{\infty} \left|1 - \frac{x}{lL}\right|^m}{\left|1 - \frac{x}{(k-1)L}\right|^m \left|1 - \frac{x}{kL}\right|^m}.$$

By applying the equality

$$\left|\frac{\sin \frac{\pi x}{L}}{\frac{\pi x}{L}}\right| = \prod_{l=-\infty, l \neq 0}^{\infty} \left|1 - \frac{x}{lL}\right|$$

and the fact that

$$\left|\frac{\sin \frac{\pi x}{L}}{\frac{\pi x}{L} \left(1 - \frac{x}{(k-1)L}\right) \left(1 - \frac{x}{kL}\right)}\right| \leq 2k,$$

we find the bound

$$|f(x)| \leq C(1+x)^{3m}(2k)^m \leq C(1+x)^{4m}.$$

On the imaginary axis we have

$$|f(iy)|^2 = \lim_{R \rightarrow \infty} \prod_{|\lambda| < R, \lambda \neq 0} \left(1 + \frac{y^2}{\lambda^2}\right).$$

We consider only the first quadrant of  $\mathbb{C}$  and let  $y > 0$ . By taking logarithms of both sides of the above equation, and by moving points of  $\Lambda$  in each interval  $\omega_k$  to the right to increase the product, we obtain

$$\begin{aligned} 2 \log |f(iy)| &= \lim_{R \rightarrow \infty} \sum_{|\lambda| < R} \log \left(1 + \frac{y^2}{\lambda^2}\right) \\ &\leq 2m \log \left(1 + \frac{y^2}{\beta^2}\right) + 2m \sum_{k=1}^{\infty} \log \left(1 + \frac{y^2}{(kL)^2}\right) \\ &\leq 2m \log \left(1 + \frac{y^2}{\beta^2}\right) + 2m \int_0^{\infty} \log \left(1 + \frac{y^2}{(kL)^2}\right) dk \\ &\leq C + 2\pi(m + \epsilon) \frac{y}{L} \end{aligned}$$

for some arbitrarily small  $\epsilon > 0$ . The letter  $\beta$  denotes the separation of  $\Lambda$  as defined in (2.2). We conclude that

$$|f(iy)| \leq C e^{\pi d_2 y}, \quad d_2 < 2a.$$

If we now apply Phragmén-Lindelöf's principle to the function of exponential type

$$F(z) = \frac{f(z)}{(1+z)^{4m}} e^{i\pi d_2 z},$$

we have that (3.6) holds in the right upper quadrant of  $\mathbb{C}$ . Similar arguments show that (3.6) holds in all four quadrants. This completes the proof of Lemma 3.6.  $\square$

We now use Lemma 3.6 to perform interpolation on  $\Lambda$ . Fix  $\epsilon$  between 0 and  $a - d_2/2$ . Choose  $h(\xi)$  in  $C_0^\infty(-\epsilon, \epsilon) \subset \mathcal{S}$  such that  $\hat{h}(0) = 1$ , where  $\mathcal{S}$  denotes the Schwarz function space. As  $\mathcal{S}$  is invariant under the Fourier transform, we have

$$|\hat{h}(x)| \leq C_l |x|^{-l}, \quad l = 1, 2, \dots$$

For every  $\lambda_j \in \Lambda$  we use Lemma 3.6 to construct an entire function  $f_j$ , using  $\lambda_j$  as the origin, such that  $f_j(\lambda_k) = 0$  for  $j \neq k$ ,  $f_j(\lambda_j) = 1$  and

$$|f_j(z)| \leq C(|z - \lambda_j| + 1)^{4m} e^{\pi d_2 |y|}.$$

Once again we apply Phragmén-Lindelöf's principle to obtain

$$|\hat{h}(z)| \leq \frac{C}{(|z| + 1)^{4m+2}} e^{2\pi\epsilon|y|}.$$

We then define

$$g_j(z) = f_j(z) \hat{h}(z - \lambda_j).$$

The function  $g_j$  satisfies  $g_j(\lambda_j) = 1$  and  $g_j(\lambda_k) = 0$  when  $j \neq k$ . Furthermore we have

$$|g_j(z)| = C(|z - \lambda_j| + 1)^{-2} e^{\pi(d_2+2\epsilon)|y|} \leq C(|z - \lambda_j| + 1)^{-2} e^{2\pi a|y|},$$

which shows that  $g_j$  is of exponential type  $< 2\pi a$ . Now given any sequence  $(w_j) \in \ell^2(\mathbb{Z})$  we construct the interpolation function

$$g(z) = \sum_{j \in \mathbb{Z}} w_j g_j(z).$$

A simple calculation shows that  $g$  is of exponential type  $< 2\pi a$ . Lastly we see that

$$\begin{aligned} \|g\|_2^2 &= \int_{\mathbb{R}} \left| \sum_j w_j g_j(x) \right|^2 dx \leq \int_{\mathbb{R}} \left( \sum_j |w_j|^2 |g_j(x)| \right) \left( \sum_j |g_j(x)| \right) dx \\ &\leq \sup_{x \in \mathbb{R}} \left( \sum_j |g_j(x)| \right) \cdot \int_{\mathbb{R}} \left( \sum_j |w_j|^2 |g_j(x)| \right) dx \\ &\leq C \cdot \sum_j \left( |w_j|^2 \int_{\mathbb{R}} |g_j(x)| dx \right) \leq \tilde{C} \|w\|_2^2 < \infty. \end{aligned}$$

Thus,  $g$  is square integrable on the real line. We have succeeded at finding a function  $g \in PW_{2\pi a}$  such that  $g(\lambda_j) = w_j$  for all  $j \in \mathbb{Z}$ . This completes the proof of Theorem 2.3 (B2).  $\square$

For an alternative approach to finding sufficient and necessary density conditions for stable interpolation sets for  $B(I)$ , the reader is encouraged to look up *Multipliers for entire functions and an interpolation problem of Beurling* by Ortega-Cerdà and Seip [12].

## 4 Landau's necessary density conditions

Theorem 2.4 was established by Henry Landau [7]. By connecting sets of stable sampling and interpolation to a specific eigenvalue problem, Landau proved that Theorem 2.4 holds for arbitrary compact sets  $K \subset \mathbb{R}^n$  in any dimension  $n$ . His arguments are quite striking in their generality and independence of dimension. Landau starts out in one dimension, where he establishes the following two theorems.

**Theorem 4.1.** [7, Theorem 1] *Let  $K$  be the union of a finite number of intervals in  $\mathbb{R}$  of total measure  $|K|$ , and let  $\Lambda$  be a set of stable sampling for  $B(K)$ . Then*

$$n^-(r) \geq |K|r - A \log^+ r - B ,$$

where  $A$  and  $B$  are constants which depend on  $K$  and  $\Lambda$ , but not on  $r$ .

**Theorem 4.2.** [7, Theorem 2] *Let  $K$  be as in Theorem 4.1, and let  $\Lambda$  be a set of stable interpolation for  $B(K)$ . Then*

$$n^+(r) \leq |K|r + A \log^+ r + B ,$$

where  $A$  and  $B$  are constants which depend on  $K$  and  $\Lambda$ , but not on  $r$ .

The functions  $n^-(r)$  and  $n^+(r)$  in the theorems above denote the smallest and largest number of elements of  $\Lambda$  to be found in an interval of length  $r$ .

Following the proofs of Theorems 4.1 and 4.2, Landau argues that the definition of density in higher dimension presented in equation (2.10) is meaningful and unambiguous. With this in place he proceeds to prove Theorem 2.4, bounding the density of sampling and interpolation sets in any  $n$ -dimensional space  $\mathbb{R}^n$ .

In this section we present the proofs of Theorems 4.1 and 4.2. We follow the proof given by Landau rather closely, deviating little from his line of arguments. The higher dimensional generalization is omitted.

### 4.1 An eigenvalue problem

We make a few preliminary remarks on notation. Recall the Fourier transform on  $L^2(\mathbb{R}^n)$  given in (2.1)

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx ,$$

to which we have now assigned the new notation  $\mathcal{F}$ . Likewise, we denote by  $\mathcal{F}^{-1}$  the inverse operator

$$\mathcal{F}^{-1} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx .$$

If  $P$  is a set in  $\mathbb{R}^n$ , we denote by  $\chi_P$  both the characteristic function of  $P$  and the operator in  $L^2(\mathbb{R}^n)$  defined by

$$\chi_P f = \chi_P(x) f(x) .$$

Let  $Q$  and  $S$  be subsets of  $\mathbb{R}^n$ . Denote by  $B(S)$  the set of functions in  $L^2(\mathbb{R}^n)$  whose Fourier transforms are supported on  $S$  and let  $D(Q) \subset L^2(\mathbb{R}^n)$  be the set of functions supported on  $Q$ . Let  $D_Q$  and  $B_S$  denote the orthogonal projections of  $L^2(\mathbb{R}^n)$  onto  $D(Q)$  and  $B(S)$ , respectively. These projections are given explicitly by

$$B_S = \mathcal{F}^{-1} \chi_S \mathcal{F} \quad (4.1)$$

and

$$D_Q = \chi_Q . \quad (4.2)$$

The following lemma lists elementary properties of the operator  $B_S D_Q B_S$ .

**Lemma 4.3.** [7, Lemma 1] *If the sets  $S$  and  $Q$  have finite measures, then the bounded self-adjoint positive operator  $B_S D_Q B_S$  is compact. Denoting its eigenvalues, arranged in nonincreasing order, by  $\lambda_k(S, Q)$ ,  $k = 0, 1, \dots$ , we have for all  $k$*

- (i)  $\lambda_k(S, Q) = \lambda_k(S + \sigma, Q + \tau) = \lambda_k(\alpha S, \alpha^{-1} Q)$ , for any  $\sigma, \tau \in \mathbb{R}^n$  and  $\alpha > 0$ ,
- (ii)  $\lambda_k(S, Q) = \lambda_k(Q, S)$ ,
- (iii)  $\sum_k \lambda_k(S, Q) = |S| \cdot |Q|$ , where  $|\cdot|$  denotes Lebesgue measure,
- (iv)  $\sum_k \lambda_k^2(S, Q) \geq \sum_k \lambda_k^2(S, Q_1) + \sum_k \lambda_k^2(S, Q_2)$ , if  $Q = Q_1 \cup Q_2$  with  $Q_1$  and  $Q_2$  disjoint,
- (v)  $\sum_k \lambda_k^2(S, Q) \geq \left\{ sq - \frac{2}{\pi^2} \log^+ sq - 2 \left( \frac{2}{\pi} \right)^2 \right\}^n$ , if  $S$  and  $Q$  are cubes with edges parallel to the coordinate axes of volumes  $s^n$  and  $q^n$  respectively,
- (vi)  $\lambda_k(S, Q_1) \leq \lambda_k(S, Q_2)$  if  $Q_1 \subset Q_2$ ,
- (vii)  $\lambda_k(S, Q) \leq \sup_{f \in B(S), f \perp \mathcal{C}_k} \|D_Q f\|_2^2 / \|f\|_2^2$ , where  $\mathcal{C}_k$  is any  $k$ -dimensional subspace of  $L^2(\mathbb{R}^n)$ ,
- (viii)  $\lambda_{k-1}(S, Q) \geq \inf_{f \in \mathcal{C}_k} \|D_Q f\|_2^2 / \|f\|_2^2$ , where  $\mathcal{C}_k$  is any  $k$ -dimensional subspace of  $B(S)$ .

*Proof.* As projections are bounded by 1, self-adjoint and idempotent, we have

$$(B_S D_Q B_S f, f) = \|D_Q B_S f\|_2^2 \leq \|f\|_2^2 , \quad (4.3)$$

so  $B_S D_Q B_S$  is bounded by 1, self-adjoint and positive. By (4.1) and (4.2) we may write  $D_Q B_S$  as an integral operator

$$D_Q B_S f(x) = \int_{\mathbb{R}^n} \chi_Q(x) f(y) k(x-y) dy ,$$

where  $\mathcal{F} k$  coincides with  $\chi_S$ . The kernel  $\chi_Q(x) k(x-y)$  is square integrable by Parseval's Theorem, so  $D_Q B_S$ , as well as  $B_S D_Q B_S$  and  $D_Q B_S D_Q$ , are compact operators [13,

p. 158]. We write  $A \sim B$  if the compact operators  $A$  and  $B$  have the same nonzero eigenvalues, multiplicities included. If  $B_S D_Q B_S \varphi = \lambda \varphi$  and  $\lambda \neq 0$ , then clearly  $\varphi \in B(S)$  and accordingly  $\varphi = B_S \varphi$ . Moreover, we have that  $\|D_Q B_S \varphi\| \neq 0$ . Applying  $D_Q$  to the equation yields  $D_Q B_S D_Q (D_Q B_S) \varphi = D_Q B_S \varphi$ , and  $\lambda$  must be an eigenvalue of  $D_Q B_S D_Q$  as well. Using the same argument in the opposite direction shows that

$$B_S D_Q B_S \sim D_Q B_S D_Q . \quad (4.4)$$

As  $D_Q B_S D_Q$  is a self-adjoint operator, its spectrum is real. This brings us to the relation  $D_Q B_S D_Q \sim C D_Q B_S D_Q C$ , where  $C$  denotes complex conjugation. Again by (4.1) and (4.2), and by using that  $C$  and  $\chi$  commute and  $C \mathcal{F} C = \mathcal{F}^{-1}$ , we get  $C D_Q B_S D_Q C = \chi_Q \mathcal{F} \chi_S \mathcal{F}^{-1} \chi_Q$ . Since  $\mathcal{F}$  is invertible we have

$$\chi_Q \mathcal{F} \chi_S \mathcal{F}^{-1} \chi_Q \sim \mathcal{F}^{-1} \chi_Q \mathcal{F} \chi_S \mathcal{F}^{-1} \chi_Q \mathcal{F} = B_Q D_S B_Q .$$

Combining the results above we obtain  $B_S D_Q B_S \sim B_Q D_S B_Q$ , which is exactly (ii). The operator  $D_Q B_S D_Q$  may be written explicitly as

$$(D_Q B_S D_Q f)(x) = \int_{\mathbb{R}^n} \chi_Q(y) \chi_Q(x) k(x-y) f(y) dy ,$$

where  $\mathcal{F} k$  coincides with  $\chi_S$ . A change of variables in the above equation combined with (4.4) yields (i). By applying known results about square integrable kernel operators [13, p. 243-245], we obtain

$$\sum_k \lambda_k(S, Q) = \int_{\mathbb{R}^n} \chi_Q(x) k(0) dx = |Q| \cdot |S| ,$$

as claimed in (iii), and

$$\sum_k \lambda_k^2(S, Q) = \int \int_{Q \times Q} |k(x-y)|^2 dx dy . \quad (4.5)$$

Say  $Q_1$  and  $Q_2$  are disjoint subsets of  $\mathbb{R}^n$  and  $Q = Q_1 \cup Q_2$ . Then  $Q \times Q$  includes  $Q_1 \times Q_1 \cup Q_2 \times Q_2$ , and since the integrand in (4.5) is nonnegative we have (iv). We proceed to evaluate (4.5) in the special case where  $S$  and  $Q$  are cubes in  $\mathbb{R}^n$  with edges parallel to the coordinate axes and of volumes  $s^n$  and  $q^n$ , respectively. By (i) we may assume that the cubes  $S$  and  $Q$  are centered at the origin. We have

$$k(x) = \int_{\mathbb{R}^n} \chi_S(y) e^{2\pi i y \cdot x} dy = \prod_{j=1}^n \frac{\sin \pi s x_j}{\pi x_j} ,$$

where  $x = (x_1, \dots, x_n)$ . Substituting this into (4.5) yields

$$\sum_k \lambda_k^2(S, Q) = \left\{ \int_{|u| < q/2} \int_{|v| < q/2} \frac{\sin^2 \pi s(u-v)}{\pi^2 (u-v)^2} du dv \right\}^n .$$

By a change of variables, some manipulation and the identity

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi ,$$

we arrive at (v). The details of this calculation are found in Appendix C. Results by Weyl and Courant [13, p. 238] prove (vi), as  $B_S D_{Q_2} B_S$  and  $B_S D_{Q_1} B_S$  differ by the positive operator  $B_S D_{Q_2 - Q_1} B_S$  when  $Q_1 \subset Q_2$ . Furthermore, it follows from the so-called Weyl-Courant Lemma, or Min-Max Theorem, that

$$\lambda_k(S, Q) \leq \sup_{f \perp \mathcal{C}_k} (B_S D_Q B_S f, f) / \|f\|_2^2 \quad (4.6)$$

and

$$\lambda_{k-1}(S, Q) \geq \inf_{f \in \mathcal{C}_k} (B_S D_Q B_S f, f) / \|f\|_2^2 , \quad (4.7)$$

where  $\mathcal{C}_k$  is any  $k$ -dimensional subspace of  $L^2(\mathbb{R}^n)$ . With the additional restriction that  $\mathcal{C}_k \subseteq B(S)$ , we have that  $B_S f = f$ . We get

$$(B_S D_Q B_S f, f) = \|D_Q B_S f\|_2^2 = \|D_Q f\|_2^2 ,$$

and inserting this into (4.7) yields (viii). Lastly, consider the space  $B_S \mathcal{C}_k \subseteq B(S)$ , which must be of dimension  $d \leq k$ . As  $f \perp B_S \mathcal{C}_k$  if and only if  $B_S f \perp \mathcal{C}_k$ , we have that

$$\lambda_d(S, Q) \leq \sup_{B_S f \perp \mathcal{C}_k} \frac{\|D_Q B_S f\|_2^2}{\|f\|_2^2} \leq \sup_{B_S f \perp \mathcal{C}_k} \frac{\|D_Q B_S f\|_2^2}{\|B_S f\|_2^2} \leq \sup_{f \perp \mathcal{C}_k, f \in B(S)} \frac{\|D_Q f\|_2^2}{\|f\|_2^2} .$$

Thus the right hand side of (vii) is an upper bound for  $\lambda_d(S, Q) \geq \lambda_k(S, Q)$ . This completes the proof of Lemma 4.3.  $\square$

## 4.2 Connection to sets of stable sampling and interpolation

Let  $\Lambda$  be a uniformly discrete set in  $\mathbb{R}^n$ . Denote by  $n$  a counting function, which for any compact set  $I \subset \mathbb{R}^n$  returns the number of elements of  $\Lambda$  contained in  $I$ . We now make a connection between the counting function for sets of stable sampling and interpolation and certain eigenvalues discussed in the previous section.

**Lemma 4.4.** [7, Lemma 2] *Let  $S$  be a compact set and  $\Lambda$  be a set of stable sampling for  $B(S)$  with separation  $\beta > 0$  and counting function  $n$ . Let  $I$  be any compact set, and  $I_+$  be the set of points whose distance to  $I$  is less than  $\beta/2$ . Then*

$$\lambda_{n(I_+)}(S, I) \leq \gamma < 1 ,$$

where  $\gamma$  depends on  $S$  and  $\Lambda$ , but not  $I$ .

*Proof.* Let  $h(y) \in L^2(\mathbb{R}^n)$  be the function (2.8) defined in Section 2.1 which vanishes for  $|y| \geq \beta/2$  and whose Fourier transform  $\mathcal{F}h$  satisfies  $|\mathcal{F}h(x)| \geq 1$  for all  $x \in S$ . Given any  $f \in B(S)$  we construct the function

$$g(x) = \int_{\mathbb{R}^n} f(y)h(x-y)dy = \int_{|x-y| < \beta/2} f(y)h(x-y)dy, \quad (4.8)$$

which necessarily lies in  $B(S)$ . As justified in Section 2.1 we know that

$$\|g\|_2^2 \geq \|f\|_2^2 \quad (4.9)$$

and

$$|g(x)|^2 \leq \|h\|_2^2 \int_{|x-y| < \beta/2} |f(y)|^2 dy. \quad (4.10)$$

Because  $g \in B(S)$  it follows from the definition of  $\Lambda$  that

$$\|g\|_2^2 \leq M \sum_{\lambda \in \Lambda} |g(\lambda)|^2. \quad (4.11)$$

Now say  $\mathcal{C}$  is the subspace of  $L^2(\mathbb{R}^n)$  spanned by the functions  $\overline{h(\lambda - x)}$ , where  $\lambda \in \Lambda \cap I_+$ . Because these functions have disjoint supports, they must be orthogonal. Accordingly, the dimension of  $\mathcal{C}$  is  $n(I_+)$ . If  $f \in B(S)$  and  $f \perp \mathcal{C}$ , then from (4.8) we have that  $g(\lambda) = 0$  for all  $\lambda \in \Lambda \cap I_+$ . Combining (4.9), (4.10) and (4.11) we obtain

$$\begin{aligned} \|f\|_2^2 &\leq \|g\|_2^2 \leq M \sum_{\lambda \in \Lambda, \lambda \notin I_+} |g(\lambda)|^2 \\ &\leq M \|h\|_2^2 \sum_{\lambda \in \Lambda, \lambda \notin I_+} \int_{|y-\lambda| < \beta/2} |f(y)|^2 dy \\ &\leq M \|h\|_2^2 \int_{y \notin I} |f(y)|^2 dy = M \|h\|_2^2 \left[ \|f\|_2^2 - \int_I |f(y)|^2 dy \right]. \end{aligned}$$

Dividing both sides of this inequality by  $\|f\|_2^2$  and rearranging terms, we get

$$\frac{\|D_I f\|_2^2}{\|f\|_2^2} \leq 1 - M^{-1} \|h\|_2^{-2} = \gamma < 1.$$

We apply Lemma 4.3 (vii) and conclude that

$$\lambda_{n(I_+)}(S, I) \leq \gamma < 1,$$

where  $\gamma$  depends on  $S$  and  $\Lambda$  because  $M$  and  $\|h\|_2^2$  do, but  $\gamma$  does not depend on  $I$ . The proof is complete.  $\square$

**Lemma 4.5.** [7, Lemma 3] *Let  $S$  be a compact set and  $\Lambda$  a set of interpolation for  $B(S)$  with separation  $\beta$  and counting function  $n$ . Let  $I$  be any compact set, and  $I_-$  be the set of points whose distance to  $I^c$  exceeds  $\beta/2$ . Then*

$$\lambda_{n(I_-)-1}(S, I) \geq \delta > 0 ,$$

where  $\delta$  depends on  $S$  and  $\Lambda$ , but not on  $I$ .

*Proof.* Let  $\mathcal{K}(S)$  denote the orthogonal complement in  $B(S)$  of all functions that vanish on  $\Lambda$ . By Proposition 3.5 and its proof we know there exists an  $M < \infty$  such that

$$\|g\|_2^2 \leq M \sum_{\lambda \in \Lambda} |g(\lambda)|^2 \quad (4.12)$$

for all  $g \in \mathcal{K}(S)$ .

For each  $\lambda \in \Lambda$ , let  $\varphi_\lambda \in \mathcal{K}(S)$  be the function which is 1 at  $\lambda$  and 0 at all other points of  $\Lambda$ . These functions are linearly independent. We construct  $\psi_\lambda \in B(S)$  by

$$\varphi_\lambda(x) = \int_{\mathbb{R}^n} \psi_\lambda(y) h(x-y) dy , \quad (4.13)$$

where  $h$  is still the function supported on  $(-\beta/2, \beta/2)$  whose Fourier transform  $\mathcal{F}h$  satisfies  $|\mathcal{F}h(x)| \geq 1$  on  $S$ . The functions  $\psi_\lambda$  are necessarily also linearly independent. Let  $\mathcal{C}$  be the subspace of  $B(S)$  spanned by those  $\psi_\lambda$  where  $\lambda \in \Lambda \cap I_-$ . The dimension of  $\mathcal{C}$  is  $n(I_-)$ . For any function  $f \in \mathcal{C}$  we form  $g$  as in (4.8), and from (4.10) and the definition of  $I_-$  we get

$$\sum_{\lambda \in \Lambda \cap I_-} |g(\lambda)|^2 \leq \|h\|_2^2 \int_I |f(y)|^2 dy . \quad (4.14)$$

As  $g$  is a linear combination of functions  $\varphi_\lambda$  where  $\lambda \in \Lambda \cap I_-$ , we have  $g \in \mathcal{K}(S)$ . By combining (4.9), (4.12) and (4.14) we find

$$\|f\|_2^2 \leq \|g\|_2^2 \leq M \sum_{\lambda \in \Lambda} |g(\lambda)|^2 = M \sum_{\lambda \in \Lambda \cap I_-} |g(\lambda)|^2 \leq M \|h\|_2^2 \|D_I f\|_2^2 ,$$

or equivalently

$$\frac{\|D_I f\|_2^2}{\|f\|_2^2} \geq M^{-1} \|h\|_2^{-2} = \delta > 0 .$$

Applying Lemma 4.3 (viii) we conclude that

$$\lambda_{n(I_-)-1}(S, I) \geq \delta > 0 ,$$

where the constant  $\delta$  depends on  $S$  and  $\Lambda$ , but not on  $I$ . This completes the proof of Lemma 4.5.  $\square$



### 4.3 Proofs of Theorems 4.1 and 4.2

With Lemmas 4.3 through 4.5 established, we are prepared to prove Theorems 4.1 and 4.2. Put briefly, we start out by considering a given set  $\Lambda$  of either stable sampling or interpolation for the function space  $B(K)$ , where  $K$  is a finite union of intervals in  $\mathbb{R}$ . Lemmas 4.4 and 4.5 allow us to bound certain eigenvalues of the operator  $B_K D_I B_K$ . Lastly we invoke Lemma 4.3 to compare the behavior of  $\Lambda$  to that of a well known set, namely the integers  $\mathbb{Z}$ .

We make the following observation on the set  $\mathbb{Z}$ . Let  $I$  denote the unit interval centered at the origin. For any function  $f \in B(I)$  the value of  $f$  at the integer  $x = -k$  corresponds to the  $k$ th Fourier coefficient of  $\mathcal{F}f$ . By Riesz-Fischer's Theorem it follows that

$$\mathcal{F}f(x) = \lim_{n \rightarrow \infty} \sum_{k=-n}^n f(-k) e^{2\pi i k x} ,$$

and from Parseval's Theorem we get

$$\sum_{k=-\infty}^{\infty} |f(k)|^2 = \|\mathcal{F}f\|_{L^2(I)}^2 = \|f\|_2^2 .$$

Thus, the integer set  $\mathbb{Z}$  is a set of stable sampling for  $B(I)$ . Furthermore, Riesz-Fischer's Theorem claims that for any sequence  $(a_j) \in \ell^2(\mathbb{Z})$  there exists a function  $f \in L^2(I)$  such that

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k x} .$$

Again by Parseval's Theorem we have

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \|f\|_{L^2(I)}^2 ,$$

and we conclude that the integers  $\mathbb{Z}$  are also a set of stable interpolation for  $B(I)$ .

Say  $\Lambda$  is an arbitrary set of stable sampling for  $B(K)$ , where  $K$  is the union of finitely many intervals in  $\mathbb{R}$ . Recall that Theorem 4.1 claims

$$n^-(r) \geq |K|r - A \log^+ r - B ,$$

where  $n^-(r)$  is the smallest number of elements of  $\Lambda$  to be found in an interval of length  $r$ .

*Proof.* Let  $\sigma$  be a single interval of length  $r$  satisfying  $n^-(r) = n(\sigma)$ , where  $n$  is the counting function of  $\Lambda$ . Then certainly  $n(\sigma_+) \leq n(\sigma) + 2$ , and by Lemmas 4.3 (i) and 4.4 we get

$$\lambda_{n(\sigma)+2}(K, rI) \leq \lambda_{n(\sigma_+)}(K, \sigma) \leq \gamma < 1 , \quad (4.15)$$

where  $I$  is the unit interval centered at zero and  $\gamma$  is independent of  $r$ .

Say  $K$  is the union of the  $p$  disjoint intervals  $K_1, \dots, K_p$ , each with an associated length  $l_1, \dots, l_p$ , such that  $\sum_{i=1}^p l_i = |K|$ . From Lemma 4.3 (i) and (ii) it follows that

$$\lambda_k(K, rI) = \lambda_k(rI, K) = \lambda_k(I, rK) .$$

Let  $n^*$  be the counting function corresponding to  $\mathbb{Z}$ . It is easily deduced that  $n^*(rK_-) \geq r|K| - 2p$ . This follows from the fact that each disjoint interval  $rK_{i-}$  covers at least  $\lfloor rl_i \rfloor - 1 \geq rl_i - 2$  integer points, where  $\lfloor \eta \rfloor$  denotes the integer part of  $\eta$ . We apply Lemma 4.5 to obtain

$$\lambda_{\lfloor r|K| \rfloor - 2p - 1}(K, rI) \geq \lambda_{n^*(rK_-) - 1}(I, rK) \geq \delta > 0 , \quad (4.16)$$

where  $\delta$  is independent of  $r$ .

We compare the indices of the eigenvalues given in (4.15) and (4.16) by estimating the total number of eigenvalues which are not close to 0 or 1. Accordingly, we consider

$$J(K, rI) = \sum_k \lambda_k(K, rI) (1 - \lambda_k(K, rI)) .$$

By Lemma 4.3 (iii) and (iv) we have

$$J(K, rI) \leq r|K| - \sum_i \sum_k \lambda_k^2(K_i, rI) .$$

As  $K_i$  and  $rI$  are both single intervals of lengths  $l_i$  and  $r$ , respectively, it follows from Lemma 4.3 (v) that

$$J(K, rI) \leq r|K| - \sum_i rl_i + \frac{2}{\pi^2} \sum_i \log^+ rl_i + 2p \left( \frac{2}{\pi} \right)^2 = A' \log^+ r + B' ,$$

where  $A'$  and  $B'$  are constants depending only on  $K$ . If  $n(\sigma) + 2 \leq \lfloor r|K| \rfloor - 2p - 1$ , then for each index  $k$  between these two values we have

$$0 < \delta \leq \lambda_k(K, rI) \leq \gamma < 1 ,$$

and the contribution to  $J$  from each such  $\lambda_k$  is at least

$$\alpha = \min \{ \delta(1 - \delta), \gamma(1 - \gamma) \} .$$

We get

$$(\lfloor r|K| \rfloor - 2p - n(\sigma) - 2) \alpha \leq J(K, rI) \leq A' \log^+ r + B ,$$

or equivalently

$$n(\sigma) \geq r|K| - \frac{A'}{\alpha} \log^+ r - \frac{B'}{\alpha} - 2p - 3 .$$

If  $n(\sigma) + 2 > \lfloor r|K| \rfloor - 2p - 1$ , the above inequality holds trivially. Lastly, we recall the definition of  $n(\sigma)$  and set  $A = A'/\alpha$  and  $B = B'/\alpha + 2p + 3$  to obtain

$$n^-(r) \geq r|K| - A \log^+ r - B .$$

The constants  $A$  and  $B$  depend on  $K$  and  $\Lambda$ , but not on  $r$ . This completes the proof of Theorem 4.1.  $\square$

Theorem 4.2 is proved in much the same way as Theorem 4.1, but with the roles of Lemmas 4.4 and 4.5 interchanged. Theorem 4.2 states that if  $K$  is a union of finitely many intervals and  $\Lambda$  is a set of stable interpolation for  $B(K)$ , then

$$n^+(r) \leq |K|r + A \log^+ r + B ,$$

where  $A$  and  $B$  are constants independent of  $r$ . So let  $\sigma$  be an interval of length  $r$  satisfying  $n(\sigma) = n^+(r)$ . As  $\sigma$  is a single interval we have  $n(\sigma_-) \geq n(\sigma) - 2$ , and by Lemma 4.5 we get

$$\lambda_{n(\sigma)-3}(K, rI) \geq \delta > 0 .$$

As before the notation  $n^*$  is used for the counting function of  $\mathbb{Z}$ . It is easily seen that  $n^*(rK_+) \leq \lfloor r|K| \rfloor + 2p$ . Because  $\mathbb{Z}$  is a set of stable sampling for  $B(I)$ , Lemma 4.4 yields

$$\lambda_{\lfloor r|K| \rfloor + 2p}(K, rI) \leq \lambda_{n^*(rK_+)}(I, rK) \leq \gamma < 1 .$$

Again we consider the contribution to  $J(K, rI)$  of intermediate eigenvalues under the assumption  $\lfloor r|K| \rfloor + 2p \leq n(\sigma) - 3$ . We obtain

$$(n(\sigma) - 3 - \lfloor r|K| \rfloor - 2p + 1) \alpha \leq J(K, rI) \leq A' \log^+ r + B' ,$$

or

$$n(\sigma) \leq r|K| + \frac{A'}{\alpha} \log^+ r + \frac{B'}{\alpha} + 2p + 2 .$$

The above equation holds trivially in the case  $\lfloor r|K| \rfloor + 2p > n(\sigma) - 3$ . Theorem 4.2 is obtained by letting  $A = A'/\alpha$  and  $B = B'/\alpha + 2p + 2$ , and recalling that  $n(\sigma) = n^+(r)$ .  $\square$

## 5 Density of Lattice Projections

Let  $\Gamma \subset \mathbb{R}^n$  be a lattice. Let  $E^\parallel$  be a  $k$ -dimensional subspace of  $\mathbb{R}^n$  and let  $E^\perp$  be its orthogonal complement. Let  $C_L := \{x \in \mathbb{R}^k \mid |x_i| \leq L/2\}$  be a cube with side lengths  $L$  centered at the origin. We view  $C_L$  as a subset of  $E^\parallel$ . Given that it exists, we define for  $K \subset E^\perp$  the limit

$$d_K := \lim_{L \rightarrow \infty} L^{-k} (\text{number of points of } \Gamma \text{ in } K \times (C_L + a)) , \quad (5.1)$$

where  $a \in E^\parallel$ . In this section we study what restrictions must be put on the lattice  $\Gamma$  and the set  $K$  to ensure the existence of  $d_K$ .

Consider first the special case where  $E^\parallel = \mathbb{R}^{n-1}$ ,  $E^\perp = \mathbb{R}$  and  $K = I = [-\alpha, \alpha]$ . Then

$$d_I = \lim_{L \rightarrow \infty} L^{-(n-1)} (\text{number of points of } \Gamma \text{ in } I \times (C_L + a)) ,$$

if it exists, corresponds to the density defined in (2.10) of an  $(n-1)$ -dimensional simple quasicrystal  $\Lambda_\alpha$ . This example illustrates why the quantity  $d_K$  in (5.1) might be of interest. In this section we find that for lattices  $\Gamma$  projecting densely into  $E^\perp$  and Riemann

measurable, compact sets  $K$ , the limit  $d_K$  exists uniformly in  $a \in E^\parallel$ . Furthermore, the limit exists uniformly with respect to translations of  $K$  in  $E^\perp$ . Our key tool in obtaining these results will be ergodic theory, and our approach follows that of Hof [5].

## 5.1 Definitions

Let  $X$  be a compact metric space. An  $\mathbb{R}^k$ -action on  $X$  is a continuous family  $\{T_t \mid t \in \mathbb{R}^k\}$  of homeomorphisms on  $X$  such that

1.  $T_0$  is the identity map on  $X$ , and
2.  $T_s T_t = T_{s+t}$  for all  $s, t \in \mathbb{R}^k$ .

This is also known as a flow.

Equip  $X$  with the Borel sigma algebra. A function  $f$  on  $X$  is called invariant if  $f(T_t x) = f(x)$  for all  $t \in \mathbb{R}^k$  and all  $x \in X$ . A measurable set is said to be invariant if its characteristic function is invariant. A Borel measure  $\mu$  on  $X$  is invariant if  $\mu(T_t^{-1} A) = \mu(A)$  for all measurable sets  $A \subseteq X$  and all  $t \in \mathbb{R}^k$ . It is called ergodic if the measure  $\mu(A)$  of every invariant set  $A$  is either 0 or  $\mu(X)$ . In this case we say that the flow  $T_t$  is ergodic on the space  $(X, \mu)$ . It follows from the definition that ergodicity is equivalent to saying that every bounded, invariant function on  $X$  is constant almost everywhere.

Birkhoff's Ergodic Theorem [16, Theorem 1.14] states that a measure  $\mu$  is ergodic for the flow  $T_t$  if and only if every  $f \in L^1(X, \mu)$  satisfies

$$\lim_{L \rightarrow \infty} L^{-k} \int_{C_L} f(T_t x) dt = \frac{1}{\mu(X)} \int_X f d\mu \quad \text{for a.e. } x \in X. \quad (5.2)$$

It is clear that (5.2) remains true if  $C_L$  is replaced by the translated  $C_L + a$ . If the flow only admits one invariant measure, then it is called uniquely ergodic. The flow is uniquely ergodic if and only if (5.2) holds uniformly in  $x \in X$  for all continuous functions. A proof of this last claim was given by Walters for  $\mathbb{Z}$ -actions on  $X$  by iterations of a homeomorphism [16, Theorem 6.19]. Walters' proof generalizes to the case of  $\mathbb{R}^k$ -actions by simply changing every sum in the proof by an integral over  $C_L$ .

Given a flow  $T_t$  on  $X$  and a specific point  $x \in X$ , we denote by  $\text{Orb}(x)$  the set  $\{T_t x \mid t \in \mathbb{R}^k\}$ . This set is referred to as the orbit of  $x$ . If  $\text{Orb}(x)$  is dense in  $X$  for every  $x \in X$ , we say that  $T_t$  is minimal.

Let  $D := \mathbb{R}^n / \Gamma$ , and notice that  $D$  is a compact group with infinitely many representatives in  $\mathbb{R}^n$ . We later refer to these representatives as fundamental domains in  $\mathbb{R}^n$ . Denote by  $p_0$  the canonical projection of  $\mathbb{R}^n$  onto  $D$ . For  $\xi \in \mathbb{R}^n$  define  $\chi_\xi(x) = e^{2\pi i \xi \cdot x}$ . These are the characters of the group  $\mathbb{R}^n$ . The annihilator of the lattice  $\Gamma \subset \mathbb{R}^n$  is the set

$$\Gamma^* := \{\xi \in \mathbb{R}^n \mid \chi_\xi(x) = 1 \text{ for all } x \in \Gamma\}. \quad (5.3)$$

We denote this set by  $\Gamma^*$  because it is in fact the dual or reciprocal lattice of  $\Gamma$ . If  $e_1, \dots, e_n$  forms a basis for  $\Gamma$ , then a basis for  $\Gamma^*$  is given by  $e_1^*, \dots, e_n^*$  satisfying  $\langle e_j, e_i^* \rangle =$

$\delta_{ij}$ , where  $\delta_{ij}$  denotes the Kronecker delta. To ease the notation in calculations later on we make the assumption that  $\mu(D) = 1$ .

Choose  $a^{(1)}, \dots, a^{(k)}$  independent elements from  $E^\parallel$  and define an  $\mathbb{R}^k$ -action on  $D$  by

$$T_t x := p_0 \left( x + \sum_{j=1}^k t_j a^{(j)} \right). \quad (5.4)$$

This flow leaves the Lebesgue measure  $\mu$  invariant, as it is simply a translation. Notice that  $\text{Orb}(0) = p_0(E^\parallel)$ .

## 5.2 Ergodicity

The following results tie the ergodicity of the flow (5.4) to certain properties of the lattice  $\Gamma$  and its dual  $\Gamma^*$ .

**Lemma 5.1.** (*[5, Theorem 3.1]*) *The flow (5.4) is ergodic if and only if  $E^\perp \cap \Gamma^* = \{0\}$ .*

*Proof.* Let  $f$  be a measurable, invariant and bounded function on  $D$ . Then  $f \in L^2(D, \mu)$ , and its Fourier series representation

$$f(x) = \sum_{\xi \in \Gamma^*} \hat{f}_\xi \chi_\xi(x), \quad \hat{f}_\xi := \int f \overline{\chi_\xi} d\mu,$$

converges in  $L^2(D, \mu)$ . The flow (5.4) is ergodic if and only if  $f$  is almost surely constant, or equivalently if and only if  $\hat{f}_\xi = 0$  for all  $\xi \in \Gamma^* \setminus \{0\}$ . Inserting the definition in (5.4) we have

$$f(T_t x) = \sum_{\xi \in \Gamma^*} \hat{f}_\xi \chi_\xi(T_t x) = \sum_{\xi \in \Gamma^*} \hat{f}_\xi \chi_\xi \left( \sum_{j=1}^k t_j a^{(j)} \right) \chi_\xi(x).$$

Because  $f(T_t x) = f(x)$  and the Fourier series representation of  $f$  is unique, we must have

$$\hat{f}_\xi \chi_\xi \left( \sum_{j=1}^k t_j a^{(j)} \right) = \hat{f}_\xi$$

for all  $\xi \in \Gamma^*$  and all  $t \in \mathbb{R}^k$ . This last equation shows that ergodicity of (5.4) is equivalent to requiring that the function  $t \rightarrow \chi_\xi \left( \sum_{j=1}^k t_j a^{(j)} \right)$  is not identically 1 for any  $\xi \in \Gamma^* \setminus \{0\}$ . Recalling the definition  $\chi_\xi(x) = e^{2\pi i \xi \cdot x}$ , we see that this is the same as saying no  $\xi \in \Gamma^* \setminus \{0\}$  is orthogonal to  $E^\parallel$ .  $\square$

**Lemma 5.2.** (*[5, Theorem 3.2]*) *The following properties of the flow (5.4) are equivalent:*

1. *Ergodicity;*
2. *There is an  $x_0 \in D$  such that  $\text{Orb}(x_0)$  is dense;*

3. *Minimality;*

4. *Unique ergodicity.*

*Proof.* 1 implies 2. Choose an  $x_0$ , and consider  $\text{Orb}(B_r(x_0))$ , where  $B_r(x)$  is the ball with radius  $r > 0$  centered at  $x$ . The set  $\text{Orb}(B_r(x_0))$  of positive measure is clearly invariant. Thus by ergodicity we have that  $\mu(\text{Orb}(B_r(x_0))) = 1$ . Consequently, for any  $y \in D$  we can find an  $x \in \text{Orb}(x_0)$  such that  $d(x, y) < r + \epsilon/2$  for any  $\epsilon > 0$ . Since  $r$  is arbitrary, we choose  $r = \epsilon/2$ , which yields  $d(x, y) < \epsilon$  for all  $y \in D$ . This proves that  $\text{Orb}(x_0)$  is dense in  $D$ .

2 implies 3. Since  $T_t x = (T_t x_0) + x - x_0$ , the orbit of every  $x \in D$  is dense.

3 implies 1. Suppose the flow is minimal, but not ergodic. Then by the proof of Lemma 5.1 there is a nonzero  $\xi \in \Gamma^*$  such that  $\chi_\xi(\text{Orb}(0)) = 1$ . Since  $\chi_\xi$  is continuous,  $\chi_\xi$  must equal one on the closure of  $\text{Orb}(0)$ . Then clearly  $\overline{\text{Orb}(0)}$  is a proper closed subset of  $D$ . This contradicts minimality.

1 implies 4. Let  $f$  be a continuous function. The flow (5.4) is uniquely ergodic if (5.2) holds uniformly in  $x \in D$ . Since  $D$  is compact,  $f$  is uniformly continuous. Hence, for all  $\epsilon > 0$  there is a neighborhood  $V$  of zero such that  $x - y \in V$  implies  $|f(x) - f(y)| < \epsilon/2$ .

We argue that  $\cup_{t \in \mathbb{R}^k} T_t V = D$ . Choose an  $r > 0$  and a point  $x_0 \in V$  such that  $B_{2r}(x_0) \subset V$ . We know that  $\mu(\text{Orb}(B_r(x_0))) = 1$ , and therefore that  $d(x, y) < r + \epsilon$  for all  $y \in D$ , where  $x$  is some element in  $\text{Orb}(x_0)$ . Then clearly  $y \in \text{Orb}(B_{2r}(x_0)) \subset \text{Orb}(V)$  for all  $y \in D$ , and we have that  $D \subseteq \text{Orb}(V) = \cup_{t \in \mathbb{R}^k} T_t V$ . Since  $D$  is compact, we may now choose a finite number of points  $t_1, \dots, t_p$  in  $\mathbb{R}^k$  and conclude that  $D = \cup_{i=1}^p T_{t_i} V$ .

By ergodicity there is an  $x_0 \in D$  such that

$$\lim_{L \rightarrow \infty} L^{-k} \int_{C_L} f(T_t x_0) dt = \int_D f d\mu.$$

Recall that we have assumed  $\mu(D) = 1$ . As the orbit of  $x_0$  is dense by minimality, we choose  $x_i \in \text{Orb}(x_0)$  such that  $x_i \in T_{t_i} V$  for each  $i = 1, \dots, p$ . Given  $\epsilon > 0$  we may find  $L_0$  such that for all  $L > L_0$  and all  $i = 1, \dots, p$ ,

$$\left| L^{-k} \int_{C_L} f(T_t x_i) dt - \int_D f d\mu \right| < \frac{\epsilon}{2}.$$

Now choose an arbitrary  $x \in D$ . We have that  $x \in T_{t_i} V$  for some  $i = 1, \dots, p$ . Thus,

$$\begin{aligned} \left| L^{-k} \int_{C_L} f(T_t x) dt - \int_D f d\mu \right| &\leq \left| L^{-k} \int_{C_L} f(T_t x) dt - L^{-k} \int_{C_L} f(T_t x_i) dt \right| \\ &\quad + \left| L^{-k} \int_{C_L} f(T_t x_i) dt - \int_D f d\mu \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

when  $L > L_0$ . This shows that ergodicity implies unique ergodicity.

4 trivially implies 1. □

Let  $p^\parallel$  and  $p^\perp$  denote the projections of  $\mathbb{R}^n$  onto  $E^\parallel$  and  $E^\perp$ , respectively. For  $x \in \mathbb{R}^n$ , let  $x^\parallel = p^\parallel(x)$  and  $x^\perp = p^\perp(x)$  and write  $x = (x^\parallel, x^\perp)$ .

**Lemma 5.3.** ([5, Proposition 4.3]) *The following are equivalent:*

1.  $p^\perp(\Gamma)$  is dense in  $E^\perp$ ;
2. the flow (5.4) is ergodic on  $(D, \mu)$ .

*Proof.* 1 implies 2. Let  $U$  be any open set in  $D$ , and define

$$\tilde{U} = \left\{ d \in D \mid d = u + y, u \in U, y \in E^\parallel \right\} .$$

Find an open set  $U^\perp \subset E^\perp$  such that  $p_0(U^\perp) \subset \tilde{U}$ . Such a set  $U^\perp$  always exists. Now choose  $\gamma \in \Gamma$  such that  $p^\perp(\gamma) \in U^\perp$ . We get  $\gamma = p^\perp(\gamma) + y_1$  for some  $y_1 \in E^\parallel$ , and

$$0 = p_0(\gamma) = p_0(p^\perp(\gamma) + y_1) = p_0\left(p_0(p^\perp(\gamma)) + y_1\right) = p_0(u + y_2) = T_t(u) ,$$

for  $t \in \mathbb{R}^k$  satisfying  $y_2 = \sum_{j=1}^k t_j a^{(j)} \in E^\parallel$  and  $u \in U$ . Hence  $\text{Orb}(0)$  hits  $U$ . As  $U$  was arbitrary,  $\text{Orb}(0)$  must be dense in  $D$ , and the flow (5.4) is ergodic by Lemma 5.2.

2 implies 1. By Lemma 5.2 ergodicity is equivalent to minimality for the flow (5.4). Suppose that the flow (5.4) is minimal, but  $p^\perp(\Gamma)$  is not dense in  $E^\perp$ . Then there is an open set  $U \subset E^\perp$  such that  $(U + E^\parallel) \cap \Gamma = \emptyset$ . This means that  $0 \notin p_0(U + E^\parallel)$ , or equivalently

$$0 \neq T_t(p_0(u)) \quad \text{for any } u \in U \text{ or } t \in \mathbb{R}^k .$$

The orbit of 0 does not hit  $p_0(U)$ , and since  $p_0(U)$  is transverse to  $\text{Orb}(0)$  the flow cannot be minimal.  $\square$

### 5.3 Determining the limit $d_K$

Again we turn our attention to the limit  $d_K$  given in (5.1). Recall that the set  $K$  is assumed compact, and for simplicity we have fixed  $\mu(D) = 1$ . We are ready to determine the existence and size of  $d_K$  under certain restrictions on  $K$  and  $\Gamma$ .

**Theorem 5.4.** ([5, Proposition 4.1]) *Suppose  $\Gamma^* \cap E^\perp = \{0\}$ . If  $K$  is Lebesgue measurable in  $E^\perp$  then  $d_{K+v}$  exists and is equal to  $|K|$ , the Lebesgue measure of  $K$ , for almost every  $v \in E^\perp$ .*

*Proof.* Partition  $\mathbb{R}^n$  into fundamental domains, defined earlier as representatives of  $D$  in  $\mathbb{R}^n$ . Write  $K = \cup_{i \in I} K_i$ , where each  $K_i$  is the intersection of  $K$  with a fundamental domain. It is sufficient to prove the claim for one such  $K_i$ . Thus, we make the assumption that  $K$  is contained in a single fundamental domain.

The number of points  $N_L(v)$  of  $\Gamma$  in  $(K + v) \times (C_L + a)$  equals the number of points  $\{t \in (C_L + a) \mid 0 \in T_t p_0(K + v)\}$ . Let  $A = [0, \delta)^k \subset E^\parallel$ , where  $\delta > 0$  is chosen so

small that  $\mu(K \times A) = \mu(p_0[(K+v) \times A])$ . That is, we ensure that projecting the set  $(K+v) \times A$  onto  $D$  does not produce overlaps. Then

$$\begin{aligned} N_L(v) &= \int_{C_{L+a}} \delta^{-k} I_{p_0[(K+v) \times A]}(T_{-t}0) dt \\ &= \int_{C_{L+a}} \delta^{-k} I_{p_0(K \times A)}(T_{-t}x_v) dt, \end{aligned} \quad (5.5)$$

where  $x_v := p_0[(0, -v)]$  and  $I_X$  denotes the characteristic function of the set  $X$ . Because  $\Gamma^* \cap E^\perp = \{0\}$ , it follows from Lemma 5.1 that the flow  $T_t$  is ergodic. Thus, by equation (5.2) we get

$$\lim_{L \rightarrow \infty} L^{-k} \int_{C_{L+a}} I_{p_0(K \times A)}(T_{-t}x) dt = \mu(p_0(K \times A)) = |K|\delta^k \quad (5.6)$$

for almost every  $x \in D$ . We conclude that  $d_{K+v} = |K|$  for almost every  $v \in E^\perp$ . If this were not true, the limit above would either not exist or not equal  $|K|$  for all  $x = x_v$  with  $v$  in some open set in  $E^\perp$ . As varying  $a$  in (5.6) does not affect the existence or value of the limit, and as  $a$  and  $v$  are perpendicular, it would follow that (5.6) did not hold for a set in  $D$  of positive measure.  $\square$

As proved in the previous subsection, ergodicity is equivalent to unique ergodicity for the flow (5.4) in  $D$ . The unique ergodicity allows for an even stronger statement about sets  $K$  which are Riemann measurable in  $E^\perp$ .

**Theorem 5.5.** (*[5, Proposition 4.2]*) *Suppose  $E^\perp \cap \Gamma^* = \{0\}$ . If  $K$  is Riemann measurable in  $E^\perp$  then  $d_{K+v}$  exists uniformly in  $v$  and  $a$  and is equal to  $|K|$ .*

*Proof.* We argue as in the proof of Theorem 5.4, and arrive at equation (5.5). If  $K$  is Riemann measurable in  $E^\perp$ , then from the construction of  $A$  it is clear that  $p_0(K \times A)$  must be Riemann measurable in  $D$ . Hence, for every  $\epsilon > 0$  we can find continuous functions  $\phi_1$  and  $\phi_2$  on  $D$  such that

$$\phi_1(y) \leq I_{p_0(K \times A)}(y) \leq \phi_2(y) \quad \text{for all } y \in D$$

and

$$\begin{aligned} |K|\delta^k &\geq \int \phi_1 d\mu \geq |K|\delta^k - \epsilon/2 \\ |K|\delta^k &\leq \int \phi_2 d\mu \leq |K|\delta^k + \epsilon/2. \end{aligned}$$

Since  $\phi_1$  and  $\phi_2$  are continuous there exists an  $L_0$  such that

$$\left| L^{-k} \int_{C_{L+a}} \phi_1(T_{-t}x) dt - \int \phi_1 d\mu \right| < \epsilon/2$$

and

$$\left| L^{-k} \int_{C_{L+a}} \phi_2(T_{-t}x) dt - \int \phi_2 d\mu \right| < \epsilon/2$$



whenever  $L > L_0$ . Combining the above equations we get that

$$\begin{aligned} |K|\delta^k - \epsilon &\leq L^{-k} \int_{C_{L+a}} \phi_1(T_{-t}x) dt \leq L^{-k} \int_{C_{L+a}} I_{p_0(K \times A)}(T_{-t}x) dt \\ &\leq L^{-k} \int_{C_{L+a}} \phi_2(T_{-t}x) dt \leq |K|\delta^k + \epsilon \end{aligned}$$

when  $L > L_0$ . Now let  $\epsilon$  tend to zero. We find that

$$\lim_{L \rightarrow \infty} L^{-k} \int_{C_{L+a}} I_{p_0(K \times A)}(T_{-t}x) dt = |K|\delta^k$$

holds uniformly in  $x \in D$ . It follows immediately that  $d_{K+v}$  exists uniformly in  $v$  and is equal to  $|K|$ .  $\square$

Combining Lemmas 5.3 and 5.1 with Theorem 5.5 we arrive at our introductory statement, namely that for compact, Riemann measurable sets  $K \subset E^\perp$  and lattices  $\Gamma$  projecting densely into  $E^\perp$ , the limit  $d_{K+v}$  exists uniformly in  $v$  for all  $x$  and is equal to  $|K|$ . Had the assumption  $\mu(D) = 1$  not been made, we would have found that  $d_{K+v} = c(\Gamma)|K|$ , where  $c(\Gamma) = (\mu(D))^{-1}$ .

## 6 Proof of Theorem 2.6

With the theorems in Section 5 at hand, we are ready to prove Theorem 2.6. We will only be concerned with the half of Theorem 2.6 which states when a simple quasicrystal  $\Lambda_\alpha$  is a set of stable sampling for the function space  $B(K)$ . Recall that  $B(K)$  is the subspace of functions in  $L^2(\mathbb{R})$  whose Fourier transforms are supported on the compact set  $K$ . The presented proof is a detailed version of that given by Matei and Meyer [9].

Recall the definition of the dual lattice of  $\Gamma$  given in (5.3), only now the original lattice  $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$  is  $(n+1)$ -dimensional. Equivalently we could have defined the dual lattice  $\Gamma^*$  by

$$\Gamma^* := \{y \in \mathbb{R}^{n+1} \mid x \cdot y \in \mathbb{Z}, x \in \Gamma\} .$$

For an element  $\gamma \in \Gamma$  we use the notation  $\gamma = (\tilde{x}, x)$ , where  $\tilde{x} = p_1(\gamma)$  and  $x = p_2(\gamma)$  are the projections of  $\gamma$  onto  $\mathbb{R}^n$  and  $\mathbb{R}$ , respectively. The same notation is used for elements of  $\Gamma^*$ . Similarly to how the quasicrystal  $\Lambda_\alpha$  was defined in (2.11), we define the set  $M_K \subset \mathbb{R}$  by

$$M_K = \{p_2(\gamma^*) \mid \gamma^* \in \Gamma^*, p_1(\gamma^*) \in K\} . \quad (6.1)$$

The elements of  $M_K$  are sorted in increasing order, and the resulting sequence is denoted  $(m_k)_{k \in \mathbb{Z}}$ . We make the following observations. The density of  $\Lambda_\alpha$  is uniform and equal to  $c|I|$ , where  $I = [-\alpha, \alpha]$  and  $c = c(\Gamma)$ . This is a consequence of Theorem 5.5. Indeed  $p_2(\Gamma)$  is dense in  $\mathbb{R}$ , and by Lemmas 5.3 and 5.1 we have  $\mathbb{R} \cap \Gamma^* = \{0\}$ . Thus by Theorem 5.5 we get

$$d_I = \lim_{L \rightarrow \infty} L^{-n} (\text{number of points of } \Gamma \text{ in } I \times (C_L + a)) = c|I| .$$

Lastly we recall that  $d_I$ , when it exists, is equal to the density  $D(\Lambda_\alpha)$  defined in (2.10). Furthermore, by Theorem 5.5, the density of  $M_K$  is  $|K|/c$  whenever  $K$  is Riemann measurable. We know that  $\Gamma \cap \mathbb{R}^n = \{0\}$ , and consequently  $D(M_K) = |K|/c$ . We conclude that  $|K| < D(\Lambda_\alpha)$  implies  $|I| < D(M_K)$ , which will be crucial in what follows.

Given a compact set  $K$  we replace this set by a slightly larger compact set still denoted by  $K$  which is Riemann integrable and which satisfies the inequality  $|K| < D(\Lambda)$ . We define  $M_K$  as in (6.1), and state the following preliminary lemma on the sequence  $(\tilde{m}_k)_{k \in \mathbb{Z}} \subset \mathbb{R}^n$ .

**Lemma 6.1.** [9, Lemma 2.1] *The sequence  $(\tilde{m}_k)_{k \in \mathbb{Z}}$  is equidistributed on  $K$ .*

By Lemma 6.1 we mean the following. Let  $U = [a_1, b_1] \times \dots \times [a_n, b_n]$  be a cube in  $\mathbb{R}^n$ . Then for any  $r \in \mathbb{Z}$  we have

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{k=r-T}^{r+T} I_U(\tilde{m}_k) = \frac{|K \cap U|}{|K|}, \quad (6.2)$$

where  $I_U$  is the characteristic function for the set  $U$ .

*Proof.* It follows from Theorem 5.5 that

$$d_K = \lim_{L \rightarrow \infty} L^{-1} (\text{number of elements of } \Gamma^* \text{ in } K \times (C_L + a)) = c(\Gamma^*)|K|$$

and

$$d_{K \cap U} = \lim_{L \rightarrow \infty} L^{-1} (\text{number of elements of } \Gamma^* \text{ in } (K \cap U) \times (C_L + a)) = c(\Gamma^*)|K \cap U|.$$

Both limits above hold uniformly in  $a \in \mathbb{R}$ . Accordingly we have

$$\lim_{L \rightarrow \infty} \frac{(\text{number of elements of } \Gamma^* \text{ in } (K \cap U) \times (C_L + a))}{(\text{number of elements of } \Gamma^* \text{ in } K \times (C_L + a))} = \frac{|K \cap U|}{|K|}$$

uniformly in  $a$ . We can find a specific  $a \in \mathbb{R}$  such that the above equation is exactly (6.2) with some specific  $r \in \mathbb{Z}$ .  $\square$

Pick an  $f \in C_0^\infty(K)$ . An immediate consequence of Lemma 6.1 is that

$$\frac{1}{|K|} \|\hat{f}\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{k=T}^T |\hat{f}(\tilde{m}_k)|^2. \quad (6.3)$$

In fact, Lemma 6.1 implies that the sum on the right hand side in (6.3) may be replaced by  $\sum_{k=r-T}^{r+T} |\hat{f}(\tilde{m}_k)|^2$  for any  $r \in \mathbb{Z}$ . It follows that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{k=T}^T |\hat{f}(\tilde{m}_k)|^2 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left( \sup_{r \in \mathbb{Z}} \sum_{k=r-T}^{r+T} |\hat{f}(\tilde{m}_k)|^2 \right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left( \inf_{r \in \mathbb{Z}} \sum_{k=r-T}^{r+T} |\hat{f}(\tilde{m}_k)|^2 \right). \end{aligned} \quad (6.4)$$

The right hand side of (6.3) may also be replaced by

$$\frac{c}{|K|} \lim_{\epsilon \rightarrow 0} \epsilon \sum_{k \in \mathbb{Z}} |\varphi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2, \quad (6.5)$$

where  $\varphi$  is any function from the Schwartz class  $\mathcal{S}(\mathbb{R})$  with norm  $\|\varphi\|_2^2 = 1$ . This is justified as follows. We split the sum (6.5) into blocks denoted  $B_r$  of length  $1/\sqrt{\epsilon}$ . In each such block the function  $\varphi(\epsilon m_k)$  stays approximately constant. We denote by  $m_r^*$  the point in  $B_r \cap M_K$  at which  $\varphi(\epsilon m_k)$  is maximized, and note that

$$\sqrt{\epsilon} \sum_{r \in \mathbb{Z}} |\varphi(\epsilon m_r^*)|^2 \approx \epsilon \int_{\mathbb{R}} |\varphi(\epsilon x)|^2 dx = 1.$$

We use the notation  $\approx$  to indicate that the sum on the left hand side can be made arbitrarily close to 1 by choosing  $\epsilon$  sufficiently small. Using this we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\varphi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 &= \sum_{r \in \mathbb{Z}} \sum_{B_r \cap M_K} |\varphi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 \\ &\leq \left( \sup_{r \in \mathbb{Z}} \sum_{B_r \cap M_K} |\hat{f}(\tilde{m}_k)|^2 \right) \cdot \sum_{r \in \mathbb{Z}} |\varphi(\epsilon m_r^*)|^2 \\ &\leq \frac{1}{\sqrt{\epsilon}} \cdot \sup_{r \in \mathbb{Z}} \sum_{B_r \cap M_K} |\hat{f}(\tilde{m}_k)|^2. \end{aligned}$$

Recall that the uniform density of  $M_K$  equals  $|K|/c$ . Thus given any  $\delta > 0$  we can find an  $\epsilon$  such that

$$\sqrt{\epsilon} N(\epsilon) = \frac{1}{|B_r|} \sup_{r \in \mathbb{Z}} \{\text{number of elements in } B_r\} \leq \frac{|K|}{c} + \delta.$$

For simplicity let  $N(\epsilon)$  be an integer. Returning to equation (6.5) we get

$$\begin{aligned} \frac{c}{|K|} \lim_{\epsilon \rightarrow 0} \epsilon \sum_{k \in \mathbb{Z}} |\varphi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 &\leq \frac{c}{|K|} \lim_{\epsilon \rightarrow 0} \sqrt{\epsilon} N(\epsilon) \cdot \frac{1}{N(\epsilon)} \left( \sup_{r \in \mathbb{Z}} \sum_{B_r \cap M_K} |\hat{f}(\tilde{m}_k)|^2 \right) \\ &\leq \lim_{\epsilon \rightarrow 0} \frac{1}{N(\epsilon)} \left( \sup_{r \in \mathbb{Z}} \sum_{k=r}^{r+N(\epsilon)} |\hat{f}(\tilde{m}_k)|^2 \right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{k=-T}^T |\hat{f}(\tilde{m}_k)|^2. \end{aligned}$$

For the last equality we have applied a change of variables and equation (6.4). A similar argument using the infimum in (6.4) yields the opposite inequality, and we arrive at

$$\frac{1}{|K|} \|\hat{f}\|_2^2 = \frac{c}{|K|} \lim_{\epsilon \rightarrow 0} \epsilon \sum_{k \in \mathbb{Z}} |\varphi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2. \quad (6.6)$$

At this point we introduce the auxiliary function

$$F_\epsilon(t) = \sqrt{\epsilon} \sum_{k \in \mathbb{Z}} \varphi(\epsilon m_k) \hat{f}(\tilde{m}_k) \exp(2\pi i m_k t) , \quad (6.7)$$

where  $t$  is a real variable. We may choose the function  $\varphi$  such that the Fourier transform  $\hat{\varphi}$  is a positive and even function in  $\mathcal{C}_0^\infty([-1, 1])$ . Because  $|I| > D(M_K)$  we can apply Theorem 2.3 (B2) to the interval  $I$ , the set of frequencies  $M_K$  and the sum  $F_\epsilon(t)$ . We get that

$$\epsilon \sum_{k \in \mathbb{Z}} |\varphi(\epsilon m_k)|^2 \left| \hat{f}(\tilde{m}_k) \right|^2 \leq C \int_I |F_\epsilon(t)|^2 dt . \quad (6.8)$$

We want to estimate the lim sup as  $\epsilon \rightarrow 0$  of the right hand side of the above equation. Recalling the definition of  $M_K$  we have that

$$F_\epsilon(t) = \sqrt{\epsilon} \sum_{\gamma^* \in \Gamma^*} \varphi(\epsilon p_2(\gamma^*)) \hat{f}(p_1(\gamma^*)) \exp(2\pi i p_2(\gamma^*) t) .$$

We use the Poisson summation formula to rewrite  $F_\epsilon(t)$  as

$$F_\epsilon(t) = \frac{C}{\sqrt{\epsilon}} \sum_{\gamma \in \Gamma} \hat{\varphi} \left( \frac{t + p_2(\gamma)}{\epsilon} \right) f(p_1(\gamma)) , \quad (6.9)$$

where  $C$  is a normalizing constant depending only on  $\Gamma$ . We now insert  $F_\epsilon$  as given by (6.9) in

$$\limsup_{\epsilon \rightarrow 0} \int_I |F_\epsilon(t)|^2 dt . \quad (6.10)$$

Note that all terms in (6.9) for which  $|p_2(\gamma)| \geq \alpha + \epsilon$  vanish on  $I = [-\alpha, \alpha]$ . This is a consequence of  $\hat{\varphi}$  having support in  $[-1, 1]$ . Accordingly we may restrict the sum in (6.9) to the smaller set

$$\Gamma_\epsilon = \{ \gamma \in \Gamma \mid |p_2(\gamma)| \leq \alpha + \epsilon \} .$$

We introduce the notation

$$\Gamma_\epsilon^N = \{ \gamma \in \Gamma_\epsilon \mid |p_1(\gamma)| \leq N \} ,$$

and split the sum  $F_\epsilon(t)$  restricted to  $t \in I$  as follows. Let  $F_\epsilon = F_\epsilon^N + R_N$ , where

$$F_\epsilon^N(t) = \frac{C}{\sqrt{\epsilon}} \sum_{\gamma \in \Gamma_\epsilon^N} \hat{\varphi} \left( \frac{t + p_2(\gamma)}{\epsilon} \right) f(p_1(\gamma))$$

and

$$R_N(t) = \frac{C}{\sqrt{\epsilon}} \sum_{\gamma \in \Gamma_\epsilon \setminus \Gamma_\epsilon^N} \hat{\varphi} \left( \frac{t + p_2(\gamma)}{\epsilon} \right) f(p_1(\gamma)) .$$

By applying the triangle inequality and the restriction  $0 < \epsilon \leq 1$  we find the bound  $\|R_N\|_2 \leq \epsilon_N \|\hat{\varphi}\|_2$ , where

$$\epsilon_N = C \sum_{\Gamma_1 \setminus \Gamma_1^N} |f(p_1(\gamma))| .$$

We argue that  $\epsilon_N$  tends to 0 when  $N \rightarrow \infty$ . As the function  $f$  belongs to the Schwarz class  $\mathcal{S}$ , we have

$$\sum_{\gamma \in \Gamma_1 \setminus \Gamma_1^N} |f(p_1(\gamma))| \leq C(L) \sum_{\gamma \in \Gamma_1 \setminus \Gamma_1^N} (1 + |p_1(\gamma)|)^{-L} ,$$

for any  $L \in \mathbb{N}$ . The set  $p_1(\Gamma_1)$  is uniformly sparse in  $\mathbb{R}^n$ . Thus, we construct a ball of positive volume around each point in this set in such a way that all balls are pairwise disjoint. Split  $\mathbb{R}^n$  into volumes divided by successive shells of radii  $2^m$ ,  $m = 1, 2, \dots$ . In each such section we can maximally fit on the order of  $2^{m \cdot n}$  balls of positive measure. Assume for simplicity that  $N = 2^M$  for some integer  $M$ . We get

$$\sum_{\gamma \in \Gamma_1 \setminus \Gamma_1^N} |f(p_1(\gamma))| \leq C_n \sum_{m=M}^{\infty} 2^{m \cdot n} (1 + 2^m)^{-L} < \infty ,$$

whenever  $L \geq n + 1$ . This shows that  $\epsilon_N$  is the tail of a convergent sum, so  $\epsilon_N \rightarrow 0$  as  $N \rightarrow \infty$ . We proceed to bound the norm of  $F_\epsilon^N$ . The finite set of points  $p_2(\Gamma_\epsilon^N)$  are separated in  $\mathbb{R}$  by a distance greater than  $\beta_N > 0$ . If  $\epsilon$  is chosen such that  $0 < \epsilon < \beta_N/2$ , then the different terms of  $F_\epsilon^N$  have disjoint supports. It follows that

$$\|F_\epsilon^N\|_{L^2(I)} \leq \sigma(N, \epsilon) \|\hat{\varphi}\|_2 ,$$

where

$$\sigma^2(N, \epsilon) = C^2 \sum_{\gamma \in \Gamma_\epsilon^N} |f(p_1(\gamma))|^2 .$$

When  $\epsilon$  is sufficiently small,  $\Gamma_\epsilon^N = \Gamma_0^N$  and  $\sigma(N, \epsilon) = \sigma(N, 0)$ . We have that

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \int_I |F_\epsilon(t)|^2 dt &= \limsup_{\epsilon \rightarrow 0} \|F_\epsilon^N + R_N\|_{L^2(I)}^2 \\ &\leq \limsup_{\epsilon \rightarrow 0} \left( \|F_\epsilon^N\|_{L^2(I)}^2 + \|R_N\|_2^2 + 2 \|F_\epsilon^N\|_{L^2(I)} \|R_N\|_2 \right) \quad (6.11) \\ &\leq C^2 \sum_{\gamma \in \Gamma_0^N} |f(p_1(\gamma))|^2 + \eta_N \leq C^2 \sum_{\gamma \in \Gamma_0} |f(p_1(\gamma))|^2 + \eta_N , \end{aligned}$$

where  $\eta_N = \epsilon_N(\epsilon_N + 2\sigma(N, 0))$ . Lastly, notice that  $p_1(\Gamma_0) = \Lambda_\alpha$  and that  $\eta_N \rightarrow 0$  as we let  $N \rightarrow 0$ . Combining equations (6.6), (6.8) and (6.11) we get

$$\|f\|_2^2 \leq C \sum_{\lambda \in \Lambda_\alpha} |f(\lambda)|^2 , \quad (6.12)$$

where the constant  $C$  depends only on the lattice  $\Gamma$ .

Finally, we remove the extra assumption  $\hat{f} \in \mathcal{C}_0^\infty(K)$ , and show that (6.12) holds for any  $f \in B(K)$ . The function space  $\mathcal{C}_0^\infty(K)$  is dense in  $L^2(K)$ . Thus for any  $\epsilon > 0$  and any  $\hat{f} \in L^2(K)$  we can find  $\hat{g} \in \mathcal{C}_0^\infty(K)$  such that  $\|\hat{f} - \hat{g}\|_2 = \|f - g\|_2 < \epsilon$ . For the function  $g$  we know that

$$A \|g\|_2^2 \leq \sum_{\lambda \in \Lambda_\alpha} |g(\lambda)|^2 \leq B \|g\|_2^2$$

for constants  $A$  and  $B$ . Recall from Section 2.1 that the right inequality above holds for any function in  $B(K)$ , including  $f - g$ . We have

$$\begin{aligned} \|f\|_2 &\leq \|g\|_2 + \|f - g\|_2 \leq \left( A^{-1} \sum_{\Lambda_\alpha} |g(\lambda)|^2 \right)^{1/2} + \|f - g\|_2 \\ &\leq \left( A^{-1} \sum_{\Lambda_\alpha} |f(\lambda)|^2 \right)^{1/2} + \left( A^{-1} \sum_{\Lambda_\alpha} |g(\lambda) - f(\lambda)|^2 \right)^{1/2} + \|f - g\|_2 \\ &\leq \left( A^{-1} \sum_{\Lambda_\alpha} |f(\lambda)|^2 \right)^{1/2} + C \|f - g\|_2 \leq \left( A^{-1} \sum_{\Lambda_\alpha} |f(\lambda)|^2 \right)^{1/2} + C\epsilon. \end{aligned}$$

Letting  $\epsilon$  tend to zero and squaring both sides of the inequality we get (6.12) for any  $f \in B(K)$ . This completes the proof of the first statement in Theorem 2.6.

## 7 Further work

If the set  $\mathcal{E}(\Lambda)$  of exponentials defined in (2.4) is both a frame and a Riesz sequence for  $L^2(K)$ , then  $\mathcal{E}(\Lambda)$  is said to be a Riesz basis for  $L^2(K)$ . Say that the uniformly discrete set  $\Lambda$  has uniform density  $D(\Lambda)$ . Then by Theorem 2.4 we have  $D(\Lambda) = |K|$  for any Riesz basis of exponentials  $\mathcal{E}(\Lambda)$ .

In a recent article, Kozma and Lev [6] ask whether the exponential system corresponding to a quasicrystal is a Riesz basis in  $L^2$  on appropriate multiband sets on the circle. Their results are summarized in the following theorem.

**Theorem 7.1.** *Let  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  denote the circle group, and define the set  $\Lambda(\alpha, I)$  by*

$$\Lambda(\alpha, I) = \{n \in \mathbb{Z} \mid a \leq n\alpha \leq b\},$$

where  $I = [a, b)$  and  $\alpha$  is an irrational number.

1. *If  $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ , then the exponential system  $\mathcal{E}(\Lambda(\alpha, I))$  is a Riesz basis in  $L^2(S)$  for every set  $S \subset \mathbb{T}$ ,  $|S| = |I|$ , which is the union of finitely many disjoint intervals whose lengths belong to  $\mathbb{Z} + \alpha\mathbb{Z}$ .*
2. *If  $|I| \notin \mathbb{Z} + \alpha\mathbb{Z}$ , then  $\mathcal{E}(\Lambda(\alpha, I))$  is not a Riesz basis in  $L^2(S)$  for any  $S \subset \mathbb{T}$  which is the union of finitely many intervals.*

---

The family of sets  $S$  in Theorem 7.1 is in a sense dense in  $\mathbb{T}$ . Say  $\beta \in \mathbb{Z} + \alpha\mathbb{Z} \in (0, 1)$ , and denote by  $\mathcal{U}(\alpha)$  the collection of finite unions of intervals of lengths in  $\mathbb{Z} + \alpha\mathbb{Z}$ . Let  $U \subset \mathbb{T}$  and  $K \subset \mathbb{T}$  be respectively open and compact, such that  $K \subset U$  and  $|K| < \beta < |U|$ . Then there exists a set  $S \in \mathcal{U}(\alpha)$  such that  $K \subset S \subset U$ . With Theorem 7.1, Kozma and Lev claim that  $\mathcal{E}(\Lambda(\alpha, I))$  is a universal Riesz basis for  $\mathcal{U}(\alpha)$  whenever  $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ . The essential role played by the diophantine assumption on  $|I|$  in constructing this particular basis is somewhat surprising. Kozma and Lev close their article by presenting several open problems which might be worth looking into.

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## A Existence of $D^-(\Lambda)$ and $D^+(\Lambda)$

Let  $\Lambda \subset \mathbb{R}$  be a uniformly discrete set with separation  $\beta > 0$ , and let  $n^-(r)$  and  $n^+(r)$  denote respectively the smallest and largest number of elements of  $\Lambda$  to be found in an interval of length  $r$ . It is quite clear that  $n^-(r)$  is a superadditive function of  $r$ , whereas  $n^+(r)$  is a subadditive function of  $r$ . We claim in Section 2.2 that because of this, the limits

$$D^-(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^-(r)}{r} \quad \text{and} \quad D^+(\Lambda) = \lim_{r \rightarrow \infty} \frac{n^+(r)}{r}$$

exist.

The following lemma not only ensures the existence of  $D^-(\Lambda)$  and  $D^+(\Lambda)$ , but specifies what the limits are.

### Lemma A.1.

$$(1) \quad D^-(\Lambda) = \sup_{r \in \mathbb{R}} n^-(r)/r.$$

$$(2) \quad D^+(\Lambda) = \inf_{r \in \mathbb{R}} n^+(r)/r.$$

*Proof.* Under the assumption that  $\Lambda$  is uniformly discrete with separation  $\beta > 0$ , we have the inequality

$$0 \leq \frac{n^-(r)}{r} \leq \frac{n^+(r)}{r} \leq \frac{1}{\beta}.$$

Accordingly, the claimed limit values of  $D^-(\Lambda)$  and  $D^+(\Lambda)$  are finite and lie in the interval  $[0, 1/\beta]$ .

We proceed to prove (2). Let

$$A = \inf_{r \in \mathbb{R}} \frac{n^+(r)}{r},$$

and find an  $R_1$  such that

$$A \leq \frac{n^+(R_1)}{R_1} < A + \frac{\epsilon}{2}$$

for an arbitrary  $\epsilon > 0$ . Such an  $R_1$  exists by the definition of infimum. Given  $R_1$ , we locate

$$R_2 = \max_{0 \leq r \leq R_1} \frac{n^+(r)}{R_1}.$$

Now let  $R = 2R_1R_2/\epsilon$ . For any  $r = aR_1 + b > R$ , where  $a \in \mathbb{N}$  and  $b \in [0, R_1)$ , we have

$$\begin{aligned} \frac{n^+(r)}{r} &= \frac{n^+(aR_1 + b)}{aR_1 + b} \leq \frac{an^+(R_1) + n^+(b)}{aR_1 + b} \\ &< \frac{an^+(R_1)}{aR_1} + \frac{n^+(b)}{2R_1R_2/\epsilon} \leq \frac{n^+(R_1)}{R_1} + \frac{\epsilon}{2} < A + \epsilon, \end{aligned}$$

where the subadditivity of  $n^+(r)$  is used for the first inequality. We have found an  $R = R(\epsilon)$  such that

$$r > R \quad \Rightarrow \quad \left| \frac{n^+(r)}{r} - A \right| < \epsilon.$$

As  $\epsilon$  was arbitrary, (2) follows.

For the proof of (1), we let

$$B = \sup_{r \in \mathbb{R}} \frac{n^-(r)}{r} .$$

We can assume  $B \neq 0$ , as  $B = 0$  trivially implies  $D^-(\Lambda) = 0$ . For an arbitrary  $\epsilon > 0$ , we find an  $R_1$  such that

$$B - \frac{\epsilon}{2} < \frac{n^-(R_1)}{R_1} \leq B .$$

Let  $R = 2R_1/\beta\epsilon$ , and let  $r = aR_1 + b > R$ , where  $a \in \mathbb{N}$  and  $b \in [0, R_1)$ . This yields  $(a+1) \geq 2/\beta\epsilon \geq 2B/\epsilon$ . From the superadditivity of  $n^-(r)$ , we get

$$\begin{aligned} \frac{n^-(r)}{r} &= \frac{n^-(aR_1 + b)}{aR_1 + b} \geq \frac{an^-(R_1) + n^-(b)}{aR_1 + b} \\ &> \frac{an^-(R_1)}{(a+1)R_1} > B - \frac{B}{a+1} - \frac{\epsilon}{2} \geq B - \epsilon . \end{aligned}$$

We have found an  $R = R(\epsilon)$  such that

$$r > R \quad \Rightarrow \quad \left| \frac{n^-(r)}{r} - B \right| < \epsilon$$

As  $\epsilon$  was arbitrary, (1) follows. □

## B Proof of the inclusion $B(I) \subseteq B(I)_0^\infty$

$B(I)$  is the space of square integrable functions whose Fourier transforms are supported on  $I = (-a, a)$ .  $B(I)_0^\infty$  denotes the collection of continuous, bounded functions  $f(x)$  which tend to zero as  $|x| \rightarrow \infty$  and whose Fourier transforms are supported on  $I$ . We claim in Section 3.2 that  $B(I) \subseteq B(I)_0^\infty$ . In order to prove this, we must show that any  $f(x) \in B(I)$  is bounded, continuous and tends to zero as  $|x| \rightarrow \infty$ .

Let  $f \in B(I)$ , and write  $f$  as

$$f(x) = \int_{-a}^a \hat{f}(\xi) e^{2\pi i x \xi} d\xi .$$

Because  $\hat{f} \in L^2(I) \subseteq L^1(I)$ ,  $|f|$  is bounded by the  $L^1$ -norm of  $\hat{f}$ . Furthermore,

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq \int_{-a}^a \left| \hat{f}(\xi) (e^{2\pi i x_1 \xi} - e^{2\pi i x_2 \xi}) \right| d\xi \\ &\leq \max_{-a \leq \xi \leq a} \left| e^{2\pi i (x_1 - x_2) \xi} - 1 \right| \|\hat{f}\|_{L^1(I)} \leq \sqrt{2}\epsilon \end{aligned}$$

whenever  $|x_1 - x_2| < (\epsilon/2\pi a) \min \left\{ 1, 1/\|\hat{f}\|_{L^1(I)} \right\}$ . This shows that  $f$  is continuous.

What remains is showing that  $f$  tends to zero as  $|x| \rightarrow \infty$ . Suppose this is not the case, and assume without loss of generality that the claim fails for  $x \rightarrow +\infty$ . We can

then find a separated, increasing sequence  $x_1, x_2, \dots$  such that  $|f(x_i)| > 2\epsilon$  for all  $i \in \mathbb{N}$  and some  $\epsilon > 0$ . Denote by  $\beta$  the positive separation of  $(x_i)_{i=1}^\infty$ . By Bernstein's Theorem (3.3) we find that  $|f'(x)|$  is uniformly bounded by

$$|f'(x)| \leq 2\pi a \left\| \hat{f} \right\|_{L^1(I)} .$$

Thus, there exists an  $r \in (0, \beta/2)$  such that  $|f(y)| > \epsilon$  whenever  $|y - x_i| < r$  for some  $i \in \mathbb{N}$ . We get

$$\|f\|_2^2 \geq \sum_{i=1}^{\infty} \|f\|_{L^2(x_i-r, x_i+r)}^2 \geq \sum_{i=1}^{\infty} 2r\epsilon^2 \rightarrow \infty .$$

This is a contradiction, as we know  $f \in L^2(\mathbb{R})$ . We conclude that  $f$  must tend to zero as  $|x| \rightarrow \infty$ , and accordingly  $B(I) \subseteq B(I)_0^\infty$ .

## C Detailed calculation for Lemma 4.3 (v)

We claim in the proof of Lemma 4.3 (v) that

$$\left\{ \int_{|u| < q/2} \int_{|v| < q/2} \frac{\sin^2 \pi s(u-v)}{\pi^2(u-v)^2} dudv \right\}^n \geq \left\{ sq - \frac{2}{\pi^2} \log^+ sq - 2 \left( \frac{2}{\pi} \right)^2 \right\}^n . \quad (\text{C.1})$$

The following detailed calculation shows this is indeed true.

Using the substitution

$$\begin{aligned} x &= \pi s(u-v) \\ y &= \pi s(u+v) , \end{aligned}$$

we have

$$I = \int_{|u| < q/2} \int_{|v| < q/2} \frac{\sin^2 \pi s(u-v)}{\pi^2(u-v)^2} dudv = \frac{1}{2\pi^2} \int_{-a}^a \int_{-(a-|y|)}^{a-|y|} \frac{\sin^2 x}{x^2} dx dy ,$$

where  $a = \pi sq$ . By the identity

$$\int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx = \pi ,$$

and the symmetry of  $\sin^2 x/x^2$ , we rewrite  $I$  as

$$I = \frac{a}{\pi} - \frac{2}{\pi^2} \int_0^a \int_{a-y}^\infty \frac{\sin^2 x}{x^2} dx dy . \quad (\text{C.2})$$

The integral on the right hand side above is split at  $x = a$ . We get

$$\int_0^a \int_a^\infty \frac{\sin^2 x}{x^2} dx dy \leq \int_0^a dy \int_a^\infty \frac{1}{x^2} dx = 1 . \quad (\text{C.3})$$

For the remaining part of the integral we change the order of integration to obtain

$$\int_0^a \int_{a-y}^a \frac{\sin^2 x}{x^2} dx dy = \int_0^a \int_{a-x}^a \frac{\sin^2 x}{x^2} dy dx = \int_0^a \frac{\sin^2 x}{x} dx .$$

If we assume  $a \leq 1$ , then

$$\int_0^a \frac{\sin^2 x}{x} dx \leq \int_0^a dx = a \leq 1 .$$

If  $a > 1$ , we get

$$\int_0^a \frac{\sin^2 x}{x} dx \leq \int_0^1 dx + \int_1^a \frac{1}{x} dx = 1 + \ln a .$$

In either case, we have that

$$\int_0^a \frac{\sin^2 x}{x^2} dx \leq 1 + \ln^+ a . \quad (\text{C.4})$$

Inserting the bounds in (C.3) and (C.4) into (C.2), and recalling that  $a = \pi sq$ , we find the lower bound

$$I \geq \frac{a}{\pi} - \frac{2}{\pi^2} (2 + \ln^+ a) = sq - \frac{2}{\pi^2} \ln^+ (\pi sq) - \left(\frac{2}{\pi}\right)^2 . \quad (\text{C.5})$$

Lastly, we observe that

$$\ln^+ ab \leq \ln^+ a + \ln^+ b$$

and

$$\frac{2}{\pi^2} \ln^+ \pi \leq \left(\frac{2}{\pi}\right)^2 .$$

Applying this to (C.5), we find

$$I \geq sq - \frac{2}{\pi^2} \log^+ sq - 2 \left(\frac{2}{\pi}\right)^2 ,$$

which is equivalent to (C.1).