Decay estimation in nonlinear hyperbolic system of conservation laws

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1. Introduction

We consider the Cauchy problem
\[
\begin{align*}
\frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} &= 0, \\
u(x, 0) &= u_0(x),
\end{align*}
\]
with the additional assumptions:

(A) The total variation of \(u_0(x)\) is small.

(B) For each characteristic field, \(\nabla \lambda_i(u) \cdot r_i(u)\) vanishes at a single manifold of codimension one, which is transversal to the characteristic vector \(r_i(u)\). We denote the linear degeneracy manifold by \(LD_i \equiv \{u: \nabla \lambda_i(u) \cdot r_i(u) = 0, \ u \in \mathbb{R}^n\}\). Assume \((\nabla uu \lambda_i(u) \cdot r_i(u)) \cdot r_i(u) \neq 0\) and \(0 \in LD_i\).

The initial data of (1.1) satisfies \(u_0(x) = 0\) for \(|x| \geq N\). Here \(u \in \mathbb{R}^n\), \(f: \Omega \mapsto \mathbb{R}^n\) is a smooth vector function with \(\Omega \subset \mathbb{R}^n\) being an open set. Denote \(A(u)\) as \(Df(u)\) the \(n \times n\) Jacobian matrix of the flux function \(f\). The system (1.1) is assumed to be strictly hyperbolic, that is, for every \(u \in \Omega\), the matrix \(A(u)\) has \(n\) real distinct eigenvalues, denoted by 
\[
\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).
\]
Corresponding to these eigenvalues, there are \(n\) linearly independent right eigenvectors \(r_i(u)\) and left eigenvectors \(l_i(u)\). We normalize \(r_i(u)\) by \(\|r_i(u)\| = 1\) and \(l_i(u) \cdot r_i(u) = 1\).

For the same problem with the assumptions (A) and

(C) each characteristic field is genuinely nonlinear,

we can use a similar argument to obtain similar conclusion, see Remarks 2.1 and 4.4.

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Let us introduce some basic concepts of weak solutions to the hyperbolic conservation laws, which will be used in later discussion. The weak solution considered is defined as follows:

**Definition 1.1.** A function \( u : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \) is a weak solution of the problem \((1.1)\), if \( u \) is a bounded measurable function and
\[
\int_0^T \int_{\mathbb{R}} [u \phi_t + f(u) \phi_x] \, dx \, dt + \int u_0(x) \phi(x, 0) \, dx = 0, \tag{1.2}
\]
holds for any smooth function \( \phi \) with compact support in \( \{(x, t) \mid (x, t) \in \mathbb{R}^2\} \).

Under the assumptions (A) and (B), the global existence of weak solutions to \((1.1)\) is demonstrated by introducing the Glimm scheme or wavefront tracking algorithm and using the solutions to the Riemann problems solved by Lax as building blocks, cf. [1,3,6,15,16,18–22] and references therein. There are two main elements to consider in proving the existence theory of the Cauchy problem. One is to prove the decreasing of Glimm functional and the other one is to approximate the solution to the Cauchy problem by piecing together waves in different families. Moreover, with Liu’s entropy condition, each wave in the Riemann problem is classified as the curve consisting of all the end states that can be connected by an admissible shock and rarefaction waves.

**Lemma 1.1.** With the assumptions (A) and (B), the admissible i-th curve \( W_i(\alpha)(u_0) \) has Lipschitz continuous first order derivatives.

Let us introduce the notations to be used later. The backward and forward generalized characteristics are defined in [14]. We shall take the point of view that a left characteristic does not terminate at a contact shock, but crosses the shock curve and continues with altered speed and value as a right characteristic. We shall call the resulting piecewise linear, convex curve a backward generalized characteristic, noting that we regain the property that any point is connected by a backward generalized characteristic to the initial axis, see Fig. 4. We define a forward generalized characteristic to follow a characteristic line until it terminates at a shock, and to follow the shock curve thereafter. Define
\[
\begin{align*}
N^+(t) & = \inf \{ y : u(x, t) = 0, \forall x > y \}, \\
N^+(0) & = N, \\
N^-(t) & = \sup \{ y : u(x, t) = 0, \forall x < y \}, \\
N^-(0) & = -N.
\end{align*}
\tag{1.5}
\]

**Definition 1.2.** (See [19].) A discontinuity \((u_-, u_+)\) is admissible if
\[
\sigma_i(u_-, u_+) \leq \sigma_i(u_-, u_+), \tag{1.4}
\]
for any state \( u \) on the Hugoniot curve \( S_i(u_-) \) between \( u_- \) and \( u_+ \), where \( S_i(u_-) \equiv \{ u : \sigma_i(u_-) \leq \sigma_i(u, u_- - u) \leq \sigma_i(u_+) \}. \)

Any state \( u \) on the i-th Hugoniot curve \( S_i(u_0) \) is connected to \( u_0 \) by an i-th shock wave, if the above entropy condition is satisfied. We denote \( S_i(\alpha)(u_0) \) as the state which can be connected to \( u_0 \) by an i-th shock wave of strength \( \alpha \). Note that the shock wave described here includes the case of contact discontinuity.

Another basic wave pattern used to solve Riemann problem is called the rarefaction wave. The state \( R_i(\alpha)(u_0) \) \((i = 1, 2, \ldots, n)\) is connected to \( u_0 \) by an i-th rarefaction wave of strength \( \alpha \), if
\[
\begin{align*}
\frac{d}{d\alpha} R_i(\alpha)(u_0) & = r_i \left( R_i(\alpha)(u_0) \right), \\
R_i(\alpha)(u_0) & = u_0.
\end{align*}
\]

By implicit function theorem, the Riemann problem for general systems is solved by piecing together waves in different families. Moreover, with Liu’s entropy condition, each wave in the i-th family, called an i-wave, may be the composition of several i-th admissible shocks and rarefaction waves. In [20] the authors construct the wave curve \( W_i(\alpha)(u_0) \) as the curve consisting of all the end states that can be connected to \( u_0 \) by admissible shocks, rarefaction waves or their combination of the i-th family. Here \( \alpha \) is a non-degenerate parameter along the curve. Up to a linear transformation, this parameter can be chosen as the i-th component of \( u \), i.e. \( u^i \).

The wave curve has the following regularity result:

**Lemma 1.1.** With the assumptions (A) and (B), the admissible i-th curve \( W_i(s)(u_0) \) has Lipschitz continuous first order derivatives.
It is well known that 
\[ |N^+(t)| \leq C_1 t + N, \]  
(1.7) where $C_1$ is a constant independent of $t$. As in [17] for any given $t_0$ large enough, we denote by $Y^+_i(t)$, $Y^-_i(t)$, $i = 1, \ldots, n$, the $i$-th forward characteristics issuing from the point $(N^+(t_0), t_0)$ and $(N^-(t_0), t_0)$, respectively. It follows from the strict hyperbolicity of (1.1) that there exists $t_1$ such that $Y^+_i(t)$ intersects $Y^-_j(t)$, $i, j = 1, \ldots, n$, $i > j$, before time $t_1$. Similarly from $t_1$ we can obtain $t_2$. Therefore, if we choose $t_0$ large enough we can obtain
\[ t_2 \leq Kt_0. \]  
(1.8)

Here $K$ is a constant independent of $t$. Without any ambiguity we use $Y^+_i(t)$, $Y^-_i(t)$, $i = 1, \ldots, n$, to denote the forward $i$-th characteristics issuing from any $t_0$. $\xi_i(t; t_0)$ denotes the interval $[Y^+_i(t), Y^-_i(t)]$, while $\zeta_i(t; t_0)$ denotes the horizontal interval outside $Y^+_i(t)$ and $Y^-_i(t)$ issuing from $t_0$.

We set $X_i(t) = T.V.\{\lambda_i(u(x, t)); x \in \mathbb{R}\}$. $X(t) = \sum X_i(t)$. Let $X^+_i(t)$ and $X^-_i(t)$ denote the increasing and decreasing variation of $\lambda_i$ respectively at the fixed time $t$. It is clear that $X_i(t) = X^+_i(t) + X^-_i(t)$. In this paper we use $X^+_i(\xi_i(t; \tau))$ to denote the increasing and decreasing variation of $\lambda_i$ in $\xi_i(t; \tau)$, using $X^+_i(\zeta_i(t; \tau))$ to denote the increasing and decreasing variation of $\lambda_i$ outside $\xi_i(t; \tau)$. We use the symbols $\alpha$, $\beta$, etc. to represent both waves (in symbolic sense) and their strengths (in numerical sense) in this paper.

The main result of this paper can be stated as follows.

**Theorem 1.1.** For any time $t \geq t_2$
\[ X^+_i(\xi_i(t; t_1)) \leq O(1)Q^2(t_0), \]  
(1.9)
\[ X^+_i(t) \leq \left[ \frac{1}{4} + O(1)V(0) \right] X^+_i(\xi_i(t'; t_0)) + O(1)[Q^2(t_0) + Q(t')], \]  
(1.10)
\[ Q(t_2) \leq V(t_2)X^3(t_2) + O(1)Q(t_0)V(t_2), \]  
(1.11)

provided the total variation of the initial data $u(x, 0)$ is small enough. Here $t_1 < t' < t$, $|A|$ denotes the Lebesgue measure of set $A$, $Q(t)$ and $V(t)$ will be defined in the next section.

The remainder of the paper will be organized as follows. In the next section, we review some basic properties of the system (1.1) under assumptions (A) and (B). The new wave potential in Glimm functional is introduced, together with some preliminaries of the wave interaction estimates in the wave tracing argument. In Section 3 we give some useful decay estimates and the proof of the main Theorem 1.1. In the last section we give an application of the estimation in Section 3.

**2. Basic properties and Glimm functional**

Without ambiguity we will use $W_t(u_0)$ to denote the composite wave curve through the state $u_0$. As in [11,13], in the following discussion, we will assume that a rarefaction wave is divided into several small rarefaction shocks with strength as a pre-chosen small constant in the wave front tracking method. In this way, the shock waves and rarefaction waves can be treated in the same way and the error thus caused tends to zero as the small constant approaches zero.

**Definition 2.1.** (See [20].) Let $u_r \in W_t(u_i)$ so that $u_i$ is connected to $u_r$ by $i$-discontinuities $(u_{j-1}, u_j)$, and $i$-rarefaction waves $(u_j, u_{j+1})$, $j$ odd, $1 \leq j \leq m - 1$, $u_0 = u_l$ and $u_m = u_r$. A set of vectors $\{v_0, v_1, \ldots, v_p\}$ is a partition of $(u_l, u_r)$ if

(i) $v_0 = u_l$, $v_p = u_r$, $v^{k-1}_k \leq v^{k+1}_k$, $k = 1, 2, \ldots, p$,
(ii) $\{u_0, u_1, \ldots, u_m\} \subset [v_0, v_1, \ldots, v_p]$,
(iii) $v_k \in R_i(u_j)$, $j$ odd, if $u_j < v^{j+1}_k < u^{j+1}_j$,
(iv) $v_k \in D_i(u_{j-1}, u_j)$, $j$ odd, if $u_{j-1} < v^{j+1}_k < u^{j+1}_j$. Here
\[ D_i(u_j, u_r) = \{u: (u - u_j)\sigma(u, u_j) - \left( f(u) - f(u_j) \right) = c(u)r_i(u) \text{ for some scalar } c(u) \}. \]

Then set

(1) $y_k \equiv v_k - v_{k-1}$,
(2) $\lambda_i \equiv \lambda_i(v_{k-1})$ and $[\lambda_i]_k = [\lambda_i](v_{k-1}, v_k) = \lambda_i(v_k) - \lambda_i(v_{k-1}) > 0$ if (iii) holds,
(3) $\lambda_i \equiv \sigma(u_{j-1}, u_j)$ and $[\lambda_i]_k = [\lambda_i](v_{k-1}, v_k) = 0$ if (iv) holds.
The partition is stable under the perturbation in the following sense.

**Lemma 2.1.** (See [20].) Suppose that \( u_r \in W_1(u_l), \bar{u}_r \in W_1(\bar{u}_l) \), with \( u_i^j - u_i^l = \bar{u}_i^j - \bar{u}_i^l = \alpha > 0 \), and \( |u_i - \bar{u}_i| = \beta \). Then there exist partitions \( \{v_0, v_1, \ldots, v_p\} \) and \( \{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_p\} \) for the i-waves (\( u_i, u_r \)) and (\( \bar{u}_i, \bar{u}_r \)) respectively such that \( \bar{v}_k^i - v_0^i = v_k^i - v_0^i, \) \( k = 1, 2, \ldots, p \), and the following hold:

(i) \( \sum_{k=1}^p |y_k - \bar{y}_k| = O(1) \alpha \beta. \)

(ii) \( |\lambda_{i,k} - \bar{\lambda}_{i,k}| = O(1) \beta, k = 1, 2, \ldots, p. \)

(iii) Let \( \Theta^-(u_i, u_r) \) represent the value of \( \lambda_i \) at the right state \( u_r \) minus the wave speed of the right-most i-wave in (\( u_i, u_r \)). Similar definition holds for \( \Theta^- (u_l, u_r) \).

Then

\[
|\Theta^-(u_i, u_r) - \Theta^- (\bar{u}_i, \bar{u}_r)| + |\Theta^+(u_i, u_r) - \Theta^+ (\bar{u}_i, \bar{u}_r)| = O(1) \alpha \beta.
\]

Moreover, the index set \( \{1, 2, \ldots, p\} \) can be written as a disjointed union of subsets I, II and III such that

(iv) for \( k \in I \) corresponding to rarefaction waves, both \( v_k \) and \( \bar{v}_k \) are of type (iii) of Definition 2.1 and

\[
\sum_{k \in I} |[\lambda_{i,k} - \bar{\lambda}_{i,k}]| = O(1) \alpha \beta.
\]

(v) for \( k \in II \) corresponding to discontinuities, both \( v_k \) and \( \bar{v}_k \) are of the type (iv) of Definition 2.1.

(vi) for \( k \in III \) corresponding to mixed types, \( v_k \) and \( \bar{v}_k \) are of different types and

\[
\sum_{k \in III} |[\lambda_{i,k}] + |\bar{\lambda}_{i,k}|] = O(1) \alpha \beta.
\]

Defining a proper Glimm functional is crucial in obtaining some decay estimates which are useful in studying the large time behavior problem and \( N \)-waves, cf. [4,5,7-11,13,14,17], etc. For instance, in paper [17] the author gives the decay rates of total variation of the wave speed as \( t^{-\frac{1}{2}} \). The scalar case of this problem in genuinely nonlinear conservation laws dates back to 1975 [8] and 1965 [9].

In the problem (1.1), with initial data compactly supported, for \( t \) large enough, we know the main interaction comes from the same family, so the proper definition of the same family wave potential \( Q_i \) is crucial in the study of decay estimation. We define the following Glimm type functional:

\[
\mathcal{F}(t) \equiv V(t) + M_0 Q(t).
\]

In the above definition,

\[
V(t) = \sum \{ |\alpha| : \alpha \text{ is any wave in } u(t,x) \}, \quad Q(t) = Q_d(t) + Q_s(t),
\]

\[
Q_d(t) = \sum \{ |\alpha||\beta| : \alpha \text{ and } \beta \text{ are strengths of } \alpha\text{-th and } \beta\text{-th waves of } u(t,x), \text{ respectively. } i > j, \text{ and } \alpha \text{ lies to the right of } \beta \},
\]

\[
Q_s(t) = \sum_{i=1}^n Q_s^i,
\]

\[
Q_s^i = \sum \{ X^2(t) \{ |\alpha| \Delta \lambda_i(\beta) \} + |\beta| \Delta \lambda_i(\alpha) \} : \alpha \text{ and } \beta \text{ are two } i\text{-waves in } u(t,x) \}. \quad (2.1)
\]

Here \( M_0 \) is a suitably large constant and \( |\Delta \lambda_i(\alpha)| \) is the total variation of \( \lambda_i \) across the wave \( \alpha. \)

**Remark 2.1.** In the genuinely nonlinear case, \( Q_s^j \) is defined as

\[
Q_s^j = \sum \{ |\alpha||\beta| V(t) : \alpha \text{ and } \beta \text{ are two } i\text{-waves in } u(t,x) \}. \quad (2.2)
\]

**Theorem 2.2.** \( \mathcal{F}(t) \) is non-increasing at the times of interaction provided that \( M_0 \) is chosen large enough and the total variation of \( u(x,0) \) is suitably small.

The theorem will be proved in next section.
3. Some decay estimates

The purpose of this section is to formulate our main estimation. We show that $X^+(t)$ and $Q(t)$ can be estimated step by step. Without loss of generality we assume $(\nabla u_i \lambda_i(u) \cdot r_i(u)) \cdot r_i(u) < 0$ for $u \in LD_1$.

We review some known results for later use. The following lemma estimates the speed variations.

Lemma 3.1. (See [12].) There exists a constant $\eta_0 > 0$ such that any i-wave $\alpha = (u_i, u_r)$ with $|\alpha| < \eta_0$ satisfies

$$|\Delta \lambda_i(\alpha)| \approx (|u_i^m - u_i^r| + |\alpha|)|\alpha|,$$

where $u_m$ is any state on $Wi(\lambda_i(\alpha))$ and lies between $u_i$ and $u_r$.

Lemma 3.2. (See [12].) $u_\lambda = LD_1 \cap Wi(\lambda_i(\alpha))$, we write $u_\lambda = u_{\lambda_i}(u_i)$. Then for any $u \in R_i^+(u_i) \cup S_i^-(u_i)$ and any $u_r \in S_i(u_i)$, we have

$$\lambda_i(u) - \lambda_i(u_\lambda) = \frac{c(u_{\lambda})}{2} (\mu^1 - \mu^2)^2 + O\left[\left(\left|\mu^1 - \mu^2\right|^2 + \left|\mu^1 - \mu^2\right|^4\right)|\mu^1 - \mu^2|\right]. \tag{3.1}$$

$$\sigma_i(u_r, u_l) - \lambda_i(u_\lambda) = \frac{c(u_{\lambda})}{6} \left(\left(\mu^1 - \mu^2\right)^2 + \left(\mu^1 - \mu^2\right)^2 + (\mu^1 - \mu^2)^2 + (\mu^1 - \mu^2)^2 + (\mu^1 - \mu^2)^2 \right) + O\left(\left|\mu^1 - \mu^2\right|^3 + \left|\mu^1 - \mu^2\right|^3\right), \tag{3.2}$$

where $c(u_{\lambda}) = (\nabla u_i \lambda_i(u_l) \cdot r_i(u_{\lambda})) \cdot r_i(u_{\lambda}) \neq 0$.

Lemma 3.3. (See [13].) Suppose that $(u_i, u_r)$ is a left contact i-shock, and $u_\lambda \in LD_1 \cap Wi(\lambda_i(\alpha))$, then $u_{\lambda_i} - u_i \to 0$ as $u_i \to u_r \to 0$.

Lemma 3.4. (See [13].) Suppose that $(u_i, u_r)$ is a composite i-wave. Furthermore, $u_l$ is related to $u_i$ by an i-th rarefaction $(u_l, u_m)$ and a left contact i-shock $(u_m, u_r)$. We set $u_l = Wi_1(\alpha)(u_l) = S_i(\alpha - \alpha_i) \circ R_i(\alpha_i)(u_l)$, where $\alpha_i = u_m - u_l, \alpha_r = u_r - u_m, \alpha = u_i - u_r$. Take $u_\lambda \in LD_1 \cap Wi(\lambda_i(\alpha))$. Then $\frac{d\lambda_i}{d\alpha} \mid_{Wi(\alpha) = u_\lambda} = -\frac{1}{2}$.

Lemma 3.5. With the same assumption as in Lemma 3.4 we can obtain

$$\frac{d\lambda_i(R_i(\alpha))}{d\alpha} \mid_{Wi(\alpha) = u_\lambda} = \frac{1}{4} \frac{d\lambda_i(W_i(\alpha))}{d\alpha} \mid_{Wi(\alpha) = u_\lambda}. \tag{3.3}$$

Proof. In the following, for simplicity of notation, we omit the dependency of $S_i, R_i, Wi$ on $u_i$. The left contact shock satisfies the following equality:

$$\lambda_i(u_m)(u_m - u_r) - (f(u_m) - f(u_r)) = 0. \tag{3.4}$$

Simple calculation shows that

$$\nabla f(W_i(\alpha)) W_i(\alpha) = \frac{d\eta_i}{d\alpha} \nabla f(R_i(\alpha)) R_i(\alpha)$$

$$= \nabla \lambda_i \cdot r_i(R_i(\alpha)) \frac{d\eta_i}{d\alpha} (W_i(\alpha) - R_i(\alpha)) + \lambda_i(R_i(\alpha))(W_i(\alpha) - R_i(\alpha)) \frac{d\eta_i}{d\alpha} R_i(\alpha).$$

Using the above equality we can deduce

$$\frac{d\lambda_i(R_i(\alpha))}{d\alpha} = \frac{\lambda_i(W_i(\alpha)) - \lambda_i(R_i(\alpha))}{\nabla u_i \lambda_i(W_i(\alpha)) \cdot W_i(\alpha) - R_i(\alpha)} \frac{d\lambda_i(W_i(\alpha))}{d\alpha}. \tag{3.5}$$

Using a similar calculation as that of [13] and Lemma 3.3 we obtain

$$\frac{\lambda_i(W_i(\alpha)) - \lambda_i(R_i(\alpha))}{\nabla u_i \lambda_i(W_i(\alpha)) \cdot W_i(\alpha) - R_i(\alpha)} \mid_{Wi(\alpha) = u_\lambda} = \frac{1}{4}.$$

Now we are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. As we know the difference between our definition of Glimm functional and the original definition in [12] is $Q_i$. We only need to estimate the change of $F(t)$ at times of the interaction of the same family.

Suppose that there are i-waves $\alpha, \beta, \epsilon_i$ and k-waves $\delta_k, k \neq i$, as shown in Fig. 1:

$$\alpha + \beta = \gamma + \sum_{k \neq i} \delta_k.$$
Lemma 3.6. in the proof of Theorem 1.1.

Case 2: Then, 

\[ |\gamma - \alpha - \beta| + \sum_{k \neq i} |\delta_k| = O(1)|\alpha||\beta|(|\Delta \lambda_i(\alpha)| + |\Delta \lambda_i(\beta)|) \]  

and

\[ |\Delta \lambda_i(\gamma)| - |\Delta \lambda_i(\alpha)| - |\Delta \lambda_i(\beta)| = O(1)|\alpha||\beta|(|\Delta \lambda_i(\alpha)| + |\Delta \lambda_i(\beta)|). \]  

Then,

\[ \Delta Q(t) = Q(t^+) - Q(t^-) = -X^1(t^-)(|\alpha||\Delta \lambda_i(\beta)| + |\beta||\Delta \lambda_i(\alpha)|) + \sum_{\epsilon_k} |\gamma| - |\alpha| - |\beta| + \sum_{\epsilon_i} |\Delta \lambda_i(\gamma)|X^1(t+) \]

\[ - (|\Delta \lambda_i(\alpha)| + |\Delta \lambda_i(\beta)|)X^1(t^-) + |\Delta \lambda_i(\gamma)|(|\gamma|X^1(t+) - (|\alpha| + |\beta|)X^1(t-)) \]

\[ \leq - \frac{1}{2}X^1(t^-)(|\alpha||\Delta \lambda_i(\beta)| + |\beta||\Delta \lambda_i(\alpha)|). \]

Case 2: \( \alpha > 0, \beta < 0 \) and \( \gamma = \sum_{k=1}^{m-1} \gamma_k + \gamma_m > 0 \), here \( \sum_{k=1}^{m-1} \gamma_k = \gamma', \gamma_m = \gamma. \)

\[ \Delta Q(t) = Q(t^+) - Q(t^-) \]

\[ \leq -X^1(t^-)(|\alpha||\Delta \lambda_i(\beta)| + |\beta||\Delta \lambda_i(\alpha)|) + X^1(t+) \sum_{k,j=1}^{m-1} (|\gamma_k||\Delta \lambda_i(\gamma_j)| + |\gamma_j||\Delta \lambda_i(\gamma_k)|) \]

\[ + X^1(t+) \sum_{k=1}^{m-1} (|\gamma_k||\Delta \lambda_i(\gamma_m)| + |\gamma_m||\Delta \lambda_i(\gamma_k)|) \]

\[ \leq -X^1(t^-)(|\alpha||\Delta \lambda_i(\beta)| + |\beta||\Delta \lambda_i(\alpha)|) + \frac{3}{4}X^1(t+)|\beta||\Delta \lambda_i(\alpha)| + \frac{1}{2}X^1(t+)|\alpha||\Delta \lambda_i(\beta)| \]

\[ \leq 0. \]

In the above estimation we have used the results in Lemmas 3.4 and 3.5.

Finally, if \( M_0 \) is chosen large enough and the total variation of \( u_0 \) is suitably small, we obtain

\[ \Delta \mathcal{F}(t) = \mathcal{F}(t^+) - \mathcal{F}(t^-) \leq 0. \]

This ends the proof. \( \square \)

Remark 3.1. From the proof of Theorem 2.2, it is clear that \( Q(t) \) is strictly decreasing with respect to \( t \).

With the help of the Glimm functional and the above lemmas, we can obtain the following lemma, which will be used in the proof of Theorem 11.

Lemma 3.6. For any \( t \geq t_1 \), we have

\[ X_i^\pm(\xi_i(t'; t_0)) \leq O(1)Q^2(t_0), \]  

\[ X_i^+(t) \leq X_i^+(\xi_i(t'; t_0)) + O(1)Q(t_0)[Q^2(t_0) + Q(t')], \]

\[ Q(t_1) \leq Q_0 + O(1)Q(t_0)V(t_1). \]  

Here \( t_0 < t' < t \).
Proof. Clearly for $t \geq t_1$ the total amount of $i$-waves in $\xi_i(t; t_0)$ is $Q(t_0)$. With the help of Lemma 3.1 we can easily obtain (3.8). From the definition of $Q(t)$ and (3.8), we can deduce (3.10) directly. Now we demonstrate the proof of (3.9).

From (3.8) we have

$$X^+_i(t) \leq X^+_i(\xi_i(t; t_0)) + O(1)Q^2(t_0). \quad (3.11)$$

Lemma 3.5 shows that $X^+_i(\xi_i(t; t_0)) \leq X^+_i(\xi'_i(t'; t_0))$ provided that there is no wave interaction of different families. Otherwise, as in paper [13], assume that $\alpha_i = (u_i, u_m)$ is the wave of the $i$-th family, on the left of $\alpha_j = (u_m, u_r)$ ($i > j$) and the interaction $\alpha_i + \alpha_j \rightarrow \tilde{\alpha}_i + \tilde{\alpha}_j + \sum_{k \neq i, j} \delta_k$ takes place at time $t$ with the location $x$, see Fig. 2. Use the notation $\tilde{\alpha}_i \approx (\tilde{u}_i, \tilde{u}_m)$, $\tilde{\alpha}_i \approx (\tilde{u}_m, \tilde{u}_r)$.

Take $\tilde{u}_r \in W_i(\tilde{u}_m)$ such that $\alpha_i = \tilde{u}_r - \tilde{u}_m$. We know that $|\tilde{u}_r - \tilde{u}_i| = O(1)|\alpha_i| |\alpha_j|$.

Then using Lemma 2.1 to $(u_i, u_m)$ and $(\tilde{u}_m, \tilde{u}_r)$ we obtain

$$X^+_i(\xi_i(t; t_0)) \leq X^+_i(\xi'_i(t'; t_0)) + O(1)Q(t') \leq X^+_i(t') + O(1)Q(t'). \quad (3.12)$$

Hence, the combination of (3.11) and (3.12) yields (3.10). □

Proof of Theorem 1.1. With the help of strict decreasing of $Q(t)$ the proof of (1.9) and (1.11) is similar to Lemma 3.6. We will not give it here. Now we prove the estimation (1.10). Let $D(t)$ be the distance between any backward generalized characteristic $y'_1(t)$ and $y'_k(t)$ at time $t$. Divide $D(t)$ into intervals $D_i(t)$ where $\lambda_i$ is increasing.

For every $D_j$, if the characteristics extend from $t'$ to $t$ without crossing any waves, refer to $D_1$ in Fig. 3. Clearly

$$X^+_i(D_j(t)) \leq \frac{|D_j(t)|}{t - t'}. \quad (3.13)$$

If the backward characteristics do encounter waves of other families, refer to $D_2$ in Fig. 3. Lemma 2.1 shows that

$$X^+_i(D_j(t)) \leq \frac{|D_j(t)|}{t - t'} + O(1)Q(t'). \quad (3.14)$$
For the case of an $i$-rarefaction crossing an $i$-shock, refer to $D_3(t)$ in Fig. 3. From the result of Lemma 3.5 we know that
\[ X_i^+(D_3(t)) \leq \left[ \frac{1}{4} + O(1)V(0) \right] X_i^+(D_3(t')). \] (3.15)
Combining (3.13), (3.14), and (3.15) and given that rarefaction waves arising from interaction can be bounded by $O(1)Q(t')$, the proof of the lemma is completed. $\square$

**Remark 3.2.** In scalar conservation laws, for any $t' < t$, we have
\[ X^+(t) = X^+(\xi(t)) \leq \left[ \frac{\xi(t)}{t-t'} \right] + \left[ \frac{1}{4} + O(1)V(0) \right] X^+(\xi(t')). \] (3.16)
Here $\xi(t) = [Y_L(t), Y_R(t)]$, $Y_L(t)$, and $Y_R(t)$ are the forward characteristics issuing from $(-N, 0)$ and $(N, 0)$ respectively.

### 4. Application

Applying the estimation in Section 3 and the following lemma we obtain the decay rates of $X(t)$ and $Q(t)$. The rates of decay can be stated as follows.

**Theorem 4.1.** For total variation of the initial data $u(x, 0)$ sufficiently small, and $t > 0$, we have
\[ X(t) = O(1)t^{-1+\varrho}, \] (4.1)
\[ Q(t) = O(1)t^{-\frac{3}{2}+\frac{1}{2}\varrho}. \] (4.2)
Here $\varrho = \frac{4+3\varrho-\sqrt{9\varrho^2-24\varrho+4}}{6}$, $\varrho > 0$ is a small fixed constant.

In order to obtain Theorem 4.1 we first give the following lemma.

**Lemma 4.1.** Suppose that for all $t \leq T$ and $T$ large enough, we have
\[ \max\left| \lambda_i(u(\cdot, t)) - \lambda_i(0) \right| \leq Ct^{-1+\varrho}, \] (4.3)
\[ X^+(t) \leq HCt^{-1+\varrho}, \] (4.4)
\[ Q(t) \leq (HC)^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{1}{2}\varrho}, \] (4.5)
where $C < 1$ and $H$ is sufficiently large. Then for $t \leq 2T$ we have
\[ \max\left| \lambda_i(u(\cdot, t)) - \lambda_i(0) \right| \leq C't^{-1+\varrho}, \] (4.6)
\[ X^+(t) \leq HC't^{-1+\varrho}, \] (4.7)
\[ Q(t) \leq (HC')^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{1}{2}\varrho}, \] (4.8)
for $C' = (1 + H^2T^{-\varrho})C + H_2T^{-\varrho}$, where $\varrho = \frac{4+3\varrho-\sqrt{9\varrho^2-24\varrho+4}}{6}$, $\varrho > 0$ is a small constant.

**Proof.** We prove the lemma by three steps.

**Step 1.** Since $u(x, t)$ is of compact support,
\[ X^-(T) = X^+(T) + O(1)Q(\phi T), \] (4.9)
so
\[ X(T) = 2X^+(T) + O(1)Q(\phi T). \] (4.10)
Here and in the following $\phi = \frac{1}{T}$ is independent of $t$. Using Theorem 1.1 and the definition of $Q(t)$, for $T$ large enough, we have
\[ Q(T) \leq O(1)V(T)\left[ X^2(T) + Q(\phi T) \right]. \] (4.11)
From Remark 3.1, for $T \leq t \leq 2T$, we obtain
\[ Q(t) \leq Q(T) \leq (HC)^{\frac{3}{2}}(2T)^{-\frac{3}{2}+\frac{1}{2}\varrho}\left[ O(1)V(0)2^{\frac{3(1-\varrho)}{2}}(1+\phi^{-\frac{1}{2}+\frac{1}{2}\varrho}) \right]. \]
Choose $V(0)$ small enough, such that $O(1)V(0)2^{\frac{3(1-\varrho)}{2}}(1+\phi^{-\frac{1}{2}+\frac{1}{2}\varrho}) \leq 1$. Now we complete the proof of (4.8).
Step 2. Let $T_* = T_*^0 \cdot T_*^{-\beta}$, $Y'_1 = (u^1_l, u^1_r)$, $Y'_R = (u^R_l, u^R_r)$, and for $t \in [T_*, T]$.

By direct calculation, it is clear that

$$\frac{dY'_K}{dt} - \lambda_1(0) \leq O(1) Q^2(\phi T_*).$$

(4.12)

Using (1.9), Lemmas 3.1 and 3.2 we have

$$\frac{dY'_1}{dt} - \lambda_1(0) \geq \frac{1}{3}(\lambda_1(u^1_l) - \lambda_1(0)) - O(1) X^2(t) - O(1) Q^2(\phi T_*).$$

(4.13)

Thus we obtain

$$Y'_R(T) - Y'_L(T) = \int_{T_*}^T \left[ \frac{dY'_R}{dt} - \frac{dY'_L}{dt} \right] dt + Y'_R(T_*) - Y'_L(T_*) \leq \frac{1}{3} C T^\theta + KT^\theta T^{-\theta} + O(1)(HC)^3 T^\theta T^{-\theta}.$$  

(4.14)

From (1.10) for $T' \leq T$

$$X'_1(T) \leq \frac{Y'_1(T) - Y'_1(T)}{T - T'} + \frac{1}{4}(1 + O(1)V(0)) X'_1(\xi (T'; \phi T_*)) + O(1)[Q^2(\phi T_*) + Q(T')].$$

(4.15)

Taking $T' = \xi T$ in (4.15) we obtain

$$X'_1(T) \leq \frac{1}{3} C (1 - \xi)^{-1} T^{1+\theta} + K (1 - \xi)^{-1} T^{1+\theta} T^{-\theta} \left[ 1 + O(1)V(0) \right] HC(\xi T)^{-1+\theta} + O(1)(HC)^3 T^{1+\theta} (1 - \xi)^{-1} T^{-\theta}$$

$$= C(2T)^{-1+\theta} \left[ 2^{1-\theta} \left( \frac{1}{3} \right)^{-1} (1 - \xi)^{-1} T^{1+\theta} (1 - \xi)^{-1} + O(1) H^3 (1 - \xi)^{-1} T^{-\theta} \right]$$

$$+ \left[ \frac{1}{4} + O(1)V(0) \right] H \left( \frac{2}{\xi} \right)^{1-\theta}.$$

We choose $T$ large enough, $V(0)$ small enough and $\xi$ suitably close to 1 such that $(\frac{1}{4} + O(1)V(0))(\frac{2}{\xi})^{1-\theta} < \frac{1}{2}$, then, there exists $H$ satisfying $36 \left[ \frac{1}{3} (1 - \xi)^{-1} + T^{-\theta} K (1 - \xi)^{-1} \right] \leq H \leq \sqrt{\frac{(1-\xi)^{1+\theta}}{300(1T)}},$ so $H \geq 18 \left[ \frac{1}{3} (1 - \xi)^{-1} + T^{-\theta} K (1 - \xi)^{-1} + O(1) H^3 (1 - \xi)^{-1} T^{-\theta} \right]$.

From the above calculation for $t \in [T, 2T]$ we can deduce

$$X'_1(t) \leq X'_1(T) + O(1) Q(T) \leq HC(2T)^{-1+\theta},$$

(4.16)

So far, we may take $C' = C$.

Step 3. Let $T_0 = T_*^\frac{1}{2} = T \cdot T^{-\frac{1}{2}}$. With the help of Lemmas 3.1 and 3.2, by direct calculation we have

$$\frac{dY'_1(t)}{dt} - \lambda_1(0) \geq \frac{1}{3} \left[ \lambda_1(u(Y'_1(t) + t) - \lambda_1(0)) - O(1) X^2(t) + Q^2(\phi T_*)) \right]$$

$$\geq \frac{1}{3} \left[ \lambda_1(u(Y'_1(t) + t) - \lambda_1(0)) - O(1) [(HC)^3 T^{-1+\theta} + (HC)^2 T^{-\frac{1}{2}+\frac{1}{2} \theta}] \right].$$

Denote $Y(t)$ as $Y'_1(t) - Y'_R(T_0) - \lambda_1(0)(t - T_0)$. Then, from the above calculation we have

$$\frac{dY(t)}{dt} \geq \frac{1}{3} \frac{Y(t)}{t - T_0} - O(1) [(HC)^3 T^{-1+\theta} + (HC)^2 T^{-\frac{1}{2}+\frac{1}{2} \theta}].$$

(4.17)

Solving, we obtain for $t \geq T$

$$Y(t) \geq Y(T) \left[ \frac{t - T_0}{T - T_0} \right]^{\frac{1}{2}} + O(1) T^{-1+\theta}(t - T_0).$$

(4.18)
From Fig. 4 and combining (4.12), (4.13), (4.14), (4.18) we have for \( T \leq t \leq 2T \)

\[
\lambda_1(u(x,t)) - \lambda_1(0) \geq \frac{Y^*_L(t) - Y^*_R(T_0) - \lambda_1(0)(t - T_0)}{t - T_0} - O(1)Q^2(\phi T_n)
\]

\[
\geq -C(1 + O(1)T^{-\theta}t^{-1} + \epsilon) + O(1)T^{-1+\theta}
\]

\[
\geq -t^{-1+\theta}[1 + H_2T^{-\theta}C + H_2T^{-\theta}].
\]

Similarly

\[
\lambda_1(u(x,t)) - \lambda_1(0) \leq \frac{Y^*_R(t) - Y^*_L(T_0) - \lambda_1(0)(t - T_0)}{t - T_0} + O(1)Q(\phi T_n)
\]

\[
\leq t^{-1+\theta}[H_2T^{-\theta}C + H_2T^{-\theta}].
\]

For \( T \) and \( H_2 \) large enough, let \( C' = (1 + H_2T^{-\theta})C + H_2T^{-\theta} \), we complete the proof of (4.6). □

**Remark 4.2.** In the proof of Lemma 4.1 we omit some lower order items as they are absorbed into higher order ones.

**Proof of Theorem 4.1.** The proof of this theorem is the same to the paper [14], we only give an outline of the proof here.

Use Lemma 4.1, and let \( T_n = 2^nt \), \( C_0 = C \). We need to prove \( C_{n+1} = (1 + H_2T^{-\theta})C_n + H_2T^{-\theta} < 1 \). In fact,

\[
C_{n+1} + 1 = (1 + H_2T^{-\theta})(C_n + 1).
\]

(4.19)

The above equality gives \( C_n \leq C_0 + O(1)(C_0 + 1)T^{-\theta} \). Choosing \( T \) large enough and \( C_0 < 1 \), we obtain \( C_n < 1 \) for all \( n \). □

**Remark 4.3.** From the proof of Lemma 4.1 and Theorem 4.1 we can see for scalar cubic nonlinear conservation laws the decay rates can be \( X(t) = O(1)t^{-\frac{1}{2}} \), which is consistent with the result in [14].

**Remark 4.4.** For the genuinely nonlinear system of conservation laws, using the \( Q \) defined in Remark 2.1 and similar argument as in the main theorems we can deduce

\[
X^\pm(t; t_1) \leq O(1)Q(t_1) \leq O(1)Q(t_0),
\]

(4.20)

\[
X^\pm(t) \leq \frac{|\xi(t; t_0)|}{t - t'} + O(1)Q(t_0),
\]

(4.21)

\[
Q(t_2) \leq X^3(t_2) + O(1)Q(t_0)X(t_2),
\]

(4.22)

for any \( t \geq t_2 \) and sufficiently small \( V(0) \). Here \( t_1 < t' < t \).

Using the above estimation and a similar argument as in Theorem 4.1 the following decay rates can be established. For \( t > 0 \), we have the following estimations

\[
X(t) = O(1)t^{-1+\epsilon},
\]

(4.23)

\[
Q(t) = O(1)t^{-3+3\epsilon},
\]

(4.24)

provided the total variation of the initial data \( u(x,0) \) is small enough. Here \( \epsilon = \frac{1}{2} \), which is consistent with the result in [17].
References


