Adaptive Boundary Observer for Nonlinear Hyperbolic Systems: Design and Field Testing in Managed Pressure Drilling

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***Abstract*— The present paper develops an adaptive boundary observer for systems modeled by nonlinear hyperbolic PDEs with limited number of measurements. The design is based on the backstepping method and relies only on one boundary measurement. The observer gains are obtained through solving a first-order hyperbolic system of Goursat-type PDEs. The design is implemented in an estimation problem in managed pressure drilling (MPD), where a state observer and an update-law are used to estimate the annular downhole pressure and the rate of lost circulation, respectively. The design is tested against a field scale flow-loop test in Stavanger, Norway by Statoil. The results show that the state observer converges to the actual flow and pressure values and that the update-law accurately estimates the rate of lost circulation.**

I. INTRODUCTION

We consider an adaptive observer problem for systems which can be transformed into the following nonlinear hyperbolic partial differential equations:

$$w_t(x,t) \quad = \quad \Sigma(x)w_x(x,t) + f(w,x), \tag{1}$$

with boundary conditions:

$$u(0,t) \quad = \quad qv(0,t) + v_p + \theta, \tag{2}$$
$$v(1,t) \quad = \quad U(t), \tag{3}$$

where $w = [u\ v]^\intercal$ and $w : [0,1] \times [0,\infty) \to \mathbb{R}^2$. The matrix $\Sigma : [0,1] \to \mathscr{M}_{2,2}(\mathbb{R})$, where $\mathscr{M}_{2,2}(\mathbb{R})$ denotes the set of 2×2 real matrices, is given by:

$$\Sigma(x) = \begin{pmatrix} -\epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) \end{pmatrix}. \tag{4}$$

Here, $\epsilon_1(x),\ \epsilon_2(x) > 0$. Furthermore, $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ and $v_p \in \mathbb{R}$. The constant $q \neq 0$ and $U(t)$ is the actuation. The task is to estimate the state vector w and the unknown boundary parameter θ using only a boundary measurement at $x = 1$, i.e.,

$$y(t) = u(1,t). \tag{5}$$

The following assumptions are used in this paper.

Assumption 1: The nonlinear term $f(w,x)$ has an equilibrium at the origin, i.e., $f(0,x) = \frac{\partial f}{\partial w}(0,x) = 0$. Furthermore, f is twice continuously differentiable with respect to w.

Assumption 2: The transport velocities $\epsilon_1(x), \epsilon_2(x) \in \mathbb{C}^1([0,1])$.

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The state observer problem for (1) was solved for the linear case (linear f) in [1], while the state feedback stabilization problem was solved for the quasilinear case in [2].

A. Motivation and Previous Works

Many physical processes can be modeled and transformed into the hyperbolic systems (1). Some examples include road traffic [3], gas flow pipeline [4], and flow of fluids in transmission lines ([5], [6]) and in open channels [7]. Recently, for control and estimation purposes, the distributed parameter model (1)-(3) has been used to model mud flow during oil well drilling ([8], [9], [10]).

A systematic method for control and estimation problems of distributed parameter systems was previously developed in [11]. The method is called backstepping and is primarily used for nonlinear ODEs in strict-feedback form. The backstepping method has been successfully used as a tool for control design and state estimation of many types of PDEs, including with Volterra nonlinearities [12]. The method uses a change of variables by shifting the system state with a Volterra operator. Using the method of successive approximation [13] or Marcum Q-functions [14], one may find explicit expressions for the control and observer gains. The exponentially convergent observer can be combined with the backstepping controller to solve the output-feedback problem. One of the most challenging problems in the theory of distributed parameter systems is the adaptive control and observer problems. Some approaches using Lyapunov-based design and certainty equivalence-based design have been proposed to answer this problem [15].

Backstepping has found several applications in oil and gas problems, including the gas coning control ([16], [17]), the flow control in porous media [18], the slugging control [19], the lost circulation and kick problems ([20], [21]), and the heave problem ([22], [23]).

B. Contributions of this Paper

The contributions of this paper are threefold: (i) *a state observer and an update-law* for state and parameter estimation of nonlinear hyperbolic systems with a boundary measurement, (ii) *a simplified hydraulic well model* of one-phase fluid flow inside the drill pipe, and (iii) *explicit observer gains* for the simplified hydraulic well model. Furthermore, we conduct a field scale experiment to show the design is applicable for industrial uses.

C. Organization of this Paper

This paper is organized as follow. The theoretical contribution of the adaptive observer for the nonlinear hyperbolic systems (1) is presented in section II. Section III contains a brief introduction to MPD. Here, we derived a hydraulic well model with a nonlinear frictional term to model the flow and pressure of the fluid inside the well annulus. A field scale experiment setup in Statoil flow-loop test facilities is described in section IV. Results and discussions in presented in section V. Finally, section VI is the conclusions.

II. ADAPTIVE OBSERVER DESIGN

The adaptive observer design for the nonlinear hyperbolic systems (1) is constructed using the recently developed adaptive observer design for the linear hyperbolic systems in [20]. Therefore, first we briefly review the designs of the state observer and the update-law for the linear hyperbolic systems. Afterward, we show that the designs work locally for the nonlinear systems.

A. Adaptive Observer for the Linear Hyperbolic Systems

We consider the following linear hyperbolic systems:

$$\begin{aligned} w_t(x,t) &= \Sigma(x)w_x(x,t) + C(x)w(x,t), && (6)\\ u(0,t) &= qv(0,t) + v_p + \theta, && (7)\\ v(1,t) &= U(t), && (8) \end{aligned}$$

where the matrix $C(x) = \begin{pmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{pmatrix}$.

Assumption 3: $c_1(x),\ c_2(x) \in \mathbb{C}^1([0,1])$

For a collocated setup where $u(1,t)$ is measured, we design the observer as follow:

$$\begin{aligned} \hat{w}_t &= \Sigma(x)\hat{w}_x + C(x)\hat{w} + p(x)\tilde{u}(1,t), && (9)\\ \hat{u}(0,t) &= q\hat{v}(0,t) + v_p + \hat{\theta}(t), && (10)\\ \hat{v}(1,t) &= U(t), && (11) \end{aligned}$$

where $p(x) = [p_1(x) \quad p_2(x)]^\intercal$ is the observer gain to be determined. Define the error variables $\tilde{u} = u - \hat{u}$, $\tilde{v} = v - \hat{v}$, and $\tilde{\theta} = \theta - \hat{\theta}$. Forming the error systems by subtracting (9)-(11) from (6)-(8), we have:

$$\begin{aligned} \tilde{w}_t &= \Sigma(x)\tilde{w}_x + C(x)\tilde{w} - p(x)\tilde{u}(1,t), && (12)\\ \tilde{u}(0,t) &= q\tilde{v}(0,t) + \tilde{\theta}(t), && (13)\\ \tilde{v}(1,t) &= 0. && (14) \end{aligned}$$

If the unknown parameter $\theta = 0$, the following backstepping transformation:

$$\tilde{w}(x,t) = \tilde{\gamma}(x,t) - \int_x^1 P(x,\xi)\tilde{\gamma}(\xi,t)\,\mathrm{d}\xi, \quad (15)$$

where $\tilde{\gamma} = [\tilde{\alpha}\ \tilde{\beta}]^\intercal$, is used in [1] to transform the error systems (12)-(14) into an exponentially stable target systems. This means that the estimates converge to the actual values exponentially. The transformation kernel is denoted by $P(x,\xi) = \begin{pmatrix} P^{uu} & P^{uv} \\ P^{vu} & P^{vv} \end{pmatrix}$, and satisfy the following first-order hyperbolic system of Goursat-type PDEs:

$$\begin{aligned} \epsilon_1(x)P_x^{uu} + \epsilon_1(\xi)P_\xi^{uu} &= -\epsilon_1'(\xi)P^{uu} - c_1(x)P^{vu}, && (16)\\ \epsilon_1(x)P_x^{uv} - \epsilon_2(\xi)P_\xi^{uv} &= \epsilon_2'(\xi)P^{uv} - c_1(x)P^{vv}, && (17)\\ \epsilon_2(x)P_x^{vu} - \epsilon_1(\xi)P_\xi^{vu} &= \epsilon_1'(\xi)P^{vu} + c_2(x)P^{uu}, && (18)\\ \epsilon_2(x)P_x^{vv} + \epsilon_2(\xi)P_\xi^{vv} &= -\epsilon_2'(\xi)P^{vv} + c_2(x)P^{uv}, && (19) \end{aligned}$$

with boundary conditions:

$$\begin{aligned} P^{uu}(0,\xi) &= qP^{vu}(0,\xi), && (20)\\ P^{uv}(x,x) &= \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (21)\\ P^{vu}(x,x) &= -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (22)\\ P^{vv}(0,\xi) &= \frac{1}{q}P^{uv}(0,\xi). && (23) \end{aligned}$$

The kernels evolve in the triangular domain $\mathscr{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$.

The idea of the adaptive observer design is to use the transformation (15) to transform (12)-(14) into an equivalent system together with a suitable update-law of θ. It can be easily shown that the transformation (15) map the system:

$$\begin{aligned} \tilde{\gamma}_t &= \Sigma(x)\tilde{\gamma}_x + \bar{p}\tilde{\alpha}(1,t), && (24)\\ \tilde{\alpha}(0,t) &= q\tilde{\beta}(0,t) + \tilde{\theta}(t), && (25)\\ \tilde{\beta}(1,t) &= 0, && (26) \end{aligned}$$

where $\bar{p} = [-\kappa\ 0]^\intercal$, into (12)-(14) with:

$$\begin{aligned} p_1(x) &= \kappa - \epsilon_1(1)P^{uu}(x,1) - \int_x^1 \kappa P^{uu}(x,\xi)\,\mathrm{d}\xi, && (27)\\ p_2(x) &= -\epsilon_1(1)P^{vu}(x,1) - \int_x^1 \kappa P^{vu}(x,\xi)\,\mathrm{d}\xi. && (28) \end{aligned}$$

Remark that the β system in (24)-(26) is already exponentially stable. Therefore, we only consider the α system:

$$\begin{aligned} \tilde{\alpha}_t(x,t) &= -\epsilon_1(x)\tilde{\alpha}_x(x,t) - \kappa\tilde{\alpha}(1,t), && (29)\\ \tilde{\alpha}(0,t) &= \tilde{\theta}(t). && (30) \end{aligned}$$

To simplify the above system, let $\tilde{\phi}(x,t) = \tilde{\alpha}(x,t) - \tilde{\theta}(t)$, then $\tilde{\phi}_t(x,t) + \epsilon_1(x)\tilde{\phi}_x(x,t) = \tilde{\alpha}_t(x,t) - \dot{\tilde{\theta}}(t) + \epsilon_1(x)\tilde{\alpha}_x(x,t)$. Let the update-law:

$$\dot{\hat{\theta}}(t) = \kappa\tilde{\alpha}(1,t), \quad (31)$$

for $\kappa > 0$. The $\tilde{\phi}$ system is given by:

$$\begin{aligned} \tilde{\phi}_t(x,t) &= -\epsilon_1(x)\tilde{\phi}_x(x,t), && (32)\\ \tilde{\phi}(0,t) &= 0, && (33) \end{aligned}$$

and the update-law error is given by:

$$\dot{\tilde{\theta}}(t) = -\kappa\tilde{\phi}(1,t) - \kappa\tilde{\theta}(t). \quad (34)$$

Define the following Lyapunov functional:

$$V(t) = \frac{1}{2}\tilde{\theta}(t)^2 + c\int_0^1 \frac{2-x}{\epsilon_1(x)}\tilde{\phi}(x,t)^2\,\mathrm{d}x. \quad (35)$$

where $c > 0$. Computing its first derivative with respect to t along (32), (33), and (34), yields:

$$\begin{aligned} \dot{V}(t) &= -\kappa\tilde{\theta}(t)^2 - \kappa\tilde{\phi}(1,t)\tilde{\theta}(t) \\ &\quad -c\tilde{\phi}(1,t)^2 - c\int_0^1 \tilde{\phi}(x,t)^2\,\mathrm{d}x. && (36) \end{aligned}$$

If we choose c large enough such that

$$\frac{\kappa}{2}\tilde{\theta}(t)^2 + \kappa\tilde{\phi}(1,t)\tilde{\theta}(t) + c\tilde{\phi}(1,t)^2 \ge 0, \quad (37)$$

then

$$\dot{V}(t) \leq -\frac{\kappa}{2}\tilde{\theta}(t)^2 - c\int_0^1 \tilde{\phi}(x,t)^2\,\mathrm{d}x. \qquad (38)$$

Therefore, the origin of (24)-(26) is exponentially stable in the norm

$$\left(\tilde{\theta}(t)^2 + \int_0^1 \tilde{\alpha}(x,t)^2\,\mathrm{d}x + \int_0^1 \tilde{\beta}(x,t)^2\,\mathrm{d}x\right). \qquad (39)$$

Since the transformation (15) is invertible, then stability of the system (24)-(26) is translated into the stability of the error system (12)-(14). Thus, the estimated state $\hat{w}$ and parameter $\hat{\theta}$ converge to the actual values exponentially. Furthermore, since (24) is an inhomogeneous transport equation whose solution can be obtained using the method of characteristics, the existence of a unique classical solution of the observer equation (9)-(11) is guaranteed. Hence, the main result of this subsection can be stated in the following theorem.

Theorem 1: Let $P(x,\xi)$ be the solution of (16)-(23). Then for any $\hat{u}_0, \hat{v}_0 \in \mathbb{L}^2(0,1)$, system (9)-(11) with $p(x)$ and $\hat{\theta}(t)$ are given by (27)-(28) and (31), has a unique classical solution $\hat{u}(x,t), \hat{v}(x,t) \in \mathbb{C}^{1,1}((0,1)\times(0,\infty))$. Furthermore, these estimates converge exponentially to the actual values u, v, and θ.

B. Adaptive Observer for the Nonlinear Hyperbolic Systems

The nonlinear term $f(w,x)$ in (1) can be linearized around the equilibrium point $w=0$ as follow:

$$\begin{aligned} f(w,x) &= f(0,x) + \left.\frac{\partial f(w,x)}{\partial w}\right|_{w=0} w + \bar{f}_{NL}(w,x) \\ &= \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix} w + \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} w + \bar{f}_{NL}(w,x) \\ &= \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix} w + f_{NL}(w,x). \end{aligned} \qquad (40)$$

Thus, we choose:

$$C(x) = \begin{pmatrix} 0 & f_{12}(x) \\ f_{21}(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{pmatrix}, \qquad (41)$$

so that the nonlinear term $f(w,x)$ is replaced by $C(x)w + f_{NL}(w,x)$. Consequently, due to assumption 1, $f_{12}(x), f_{21}(x) \in \mathbb{C}^1([0,1])$, which is compatible with assumption 3. Remark that, since $f_{NL}(w,x)$ is twice differentiable with respect to w and once with respect to x, there exists positive constants δ_f, and a_1, a_2, and a_3, such that if $|w| \leq \delta_f$, then for any $v \in \mathbb{R}^2$, we have:

$$|f_{NL}(w,x)| + \left|\frac{\partial f_{NL}}{\partial x}(w,x)\right| \leq a_1|w|^2, \qquad (42)$$

$$\left|\frac{\partial f_{NL}}{\partial w}(w,x)\right| \leq a_2|w|, \qquad (43)$$

$$\left|\frac{\partial^2 f_{NL}}{\partial w^2}(w,x)v\right| \leq |a_3|v|. \qquad (44)$$

The nonlinear hyperbolic systems (1)-(3) become:

$$w_t = \Sigma(x)w_x + C(x)w + f_{NL}(w,x), \qquad (45)$$

$$u(0,t) = qv(0,t) + v_p + \theta, \qquad (46)$$

$$v(1,t) = U(t). \qquad (47)$$

In this case, we design the state observer as follow:

$$\begin{aligned} \hat{w}_t &= \Sigma(x)\hat{w}_x + C(x)\hat{w} + f_{NL}(\hat{w},x) \\ &\quad + p(x)\tilde{u}(1,t), \end{aligned} \qquad (48)$$

$$\hat{u}(0,t) = q\hat{v}(0,t) + v_p + \hat{\theta}(t), \qquad (49)$$

$$\hat{v}(1,t) = U(t). \qquad (50)$$

Subtracting (45)-(47) by (48)-(50), we have:

$$\begin{aligned} \tilde{w}_t &= \Sigma(x)\tilde{w}_x + C(x)\tilde{w} + f_{NL}(\tilde{w}+\hat{w},x) \\ &\quad - f_{NL}(\hat{w},x) - p(x)\tilde{u}(1,t), \end{aligned} \qquad (51)$$

$$\tilde{u}(0,t) = q\tilde{v}(0,t) + \tilde{\theta}(t), \qquad (52)$$

$$\tilde{v}(1,t) = 0. \qquad (53)$$

Define the following transformation:

$$\hat{w}(x,t) = \hat{\gamma}(x,t) + \int_0^x L(x,\xi)\hat{\gamma}(\xi,t)\,\mathrm{d}\xi, \qquad (54)$$

$$\tilde{\gamma}(x,t) = \tilde{w}(x,t) + \int_x^1 R(x,\xi)\tilde{w}(\xi,t)\,\mathrm{d}\xi. \qquad (55)$$

Furthermore, define the following functionals:

$$\mathcal{L}[\hat{\gamma}](x,t) = \hat{\gamma}(x,t) + \int_0^x L(x,y)\hat{\gamma}(y,t)\,\mathrm{d}y, \qquad (56)$$

$$\mathcal{P}[\tilde{\gamma}](x,t) = \tilde{\gamma}(x,t) - \int_x^1 P(x,y)\tilde{\gamma}(y,t)\,\mathrm{d}y, \qquad (57)$$

$$\mathcal{R}[\tilde{\gamma}](x,t) = \tilde{\gamma}(x,t) + \int_x^1 R(x,y)\tilde{\gamma}(y,t)\,\mathrm{d}y. \qquad (58)$$

Plugging (15), (54), and (55) into (51)-(53), we have:

$$\tilde{\gamma}_t = \Sigma(x)\tilde{\gamma}_x + F[\tilde{\gamma},\hat{\gamma}] + \bar{p}\tilde{\alpha}(1,t), \qquad (59)$$

$$\tilde{\alpha}(0,t) = q\tilde{\beta}(0,t) + \tilde{\theta}(t), \qquad (60)$$

$$\tilde{\beta}(1,t) = 0. \qquad (61)$$

where

$$F[\tilde{\gamma},\hat{\gamma}] = \mathcal{R}[f_{NL}(\mathcal{L}[\hat{\gamma}],\mathcal{P}[\tilde{\gamma}])] \qquad (62)$$

Assumption 4: There exists $M > 0$ such that $|\gamma| \leq M$ uniformly in $x \in [0,1]$ and $t \geq 0$.

Under this assumption, the nonlinear term becomes $F[\tilde{\gamma},\hat{\gamma}] = F[\tilde{\gamma},\gamma-\tilde{\gamma}] = F[\tilde{\gamma}]$. Notice that the dependency on $\tilde{\gamma}$ in F comes from (15), which along with the smoothness of f_{NL} implies the bound

$$|F| \leq C_0\left(|\tilde{\gamma}|^2 + \|\tilde{\gamma}\|_{\mathbb{L}^2}^2\right). \qquad (63)$$

for $C_0 > 0$. Let $\tilde{\phi}(x,t) = \tilde{\alpha}(x,t) - \tilde{\theta}(t)$, and define $\tilde{\psi} = [\tilde{\phi}\ \tilde{\beta}]^\mathsf{T}$, then (59)-(61) becomes:

$$\tilde{\psi}_t = \Sigma(x)\tilde{\psi}_x + F[\tilde{\psi},\tilde{\theta}], \qquad (64)$$

$$\tilde{\phi}(0,t) = q\tilde{\beta}(0,t), \qquad (65)$$

$$\tilde{\beta}(1,t) = 0. \qquad (66)$$

For $\tilde{\psi} \in \mathbb{H}^2([0,1])$ and positive constants b_1, b_2, b_3, b_4, b_5, and b_6, recall the following well-known inequalities:

$$\|\tilde{\psi}\|_{\mathbb{L}^1} \leq b_1\|\tilde{\psi}\|_{\mathbb{L}^2} \leq b_2\|\tilde{\psi}\|_\infty, \qquad (67)$$

$$\|\tilde{\psi}\|_\infty \leq b_3\left(\|\tilde{\psi}\|_{\mathbb{L}^2} + \|\tilde{\psi}_x\|_{\mathbb{L}^2}\right) \leq b_4\|\tilde{\psi}\|_{\mathbb{H}^1}, \qquad (68)$$

$$\|\tilde{\psi}_x\|_\infty \leq b_5\left(\|\tilde{\psi}_x\|_{\mathbb{L}^2} + \|\tilde{\psi}_{xx}\|_{\mathbb{L}^2}\right) \leq b_6\|\tilde{\psi}\|_{\mathbb{H}^2}. \qquad (69)$$

Let the Lyapunov functional as:

$$U(t)=\frac{1}{2}\tilde{\theta}(t)^2+\int_0^1\tilde{\psi}^{\intercal}(x,t)D(x,t)\tilde{\psi}(x,t)\,\mathrm{d}x, \quad (70)$$

where

$$D(x)=\begin{pmatrix} A\frac{e^{-\mu x}}{\epsilon_1(x)} & 0 \\ 0 & B\frac{e^{\mu x}}{\epsilon_2(x)} \end{pmatrix}, \quad (71)$$

and where $B=q^2A+\lambda_2$, $A=\lambda_2 e^{\mu}$, and $\mu=\lambda_1\bar{\epsilon}$, where $\bar{\epsilon}=\max_{x\in[0,1]}\left\{\frac{1}{\epsilon_1(x)},\frac{1}{\epsilon_2(x)}\right\}$, with $\lambda_1,\lambda_2>0$. Computing the first derivative of U with respect to t, and integrating by parts, yield

$$\begin{aligned}\dot{U}(t) &= -\kappa\tilde{\theta}(t)^2-\kappa\tilde{\theta}(t)\tilde{\phi}(1,t)\\ &\quad -\int_0^1\tilde{\psi}(x,t)^{\intercal}\left(D(x)\Sigma(x)\right)_x\tilde{\psi}(x,t)\,\mathrm{d}x\\ &\quad +\left[\tilde{\psi}(x,t)^{\intercal}D(x)\Sigma(x)\tilde{\psi}(x,t)\right]_0^1. \end{aligned}\quad (72)$$

Since $\left(D(x)\Sigma(x)\right)_x\geq\lambda_1 D(x)>0$, we have:

$$\dot{U}(t) \leq -\lambda_1 U(t)-2\int_0^1\tilde{\psi}(x,t)^{\intercal}D(x)F[\tilde{\psi},\theta]\,\mathrm{d}x \quad (73)$$

For $\|\tilde{\psi}\|_\infty\leq\delta_1$, where $\delta_1>0$, the last term can be estimated as follow

$$\begin{aligned}&2\int_0^1\tilde{\psi}(x,t)^{\intercal}D(x)F[\tilde{\psi},\theta]\,\mathrm{d}x \quad (74)\\ &\leq K_1\int_0^1|\psi|F[\tilde{\psi},\theta]\,\mathrm{d}x\leq K_2\|\tilde{\psi}\|_\infty\left(|\tilde{\theta}|^2+\|\tilde{\psi}\|^2_{\mathbb{L}_2}\right).\end{aligned}$$

Using (69), we have:

$$\dot{U}(t) \leq -\lambda_1 U(t)-K_1\|\tilde{\psi}_x\|_\infty U(t)+K_2U(t)^{3/2} \quad (75)$$

Denote $\tilde{\eta}=\tilde{\psi}_t$, $\tilde{\vartheta}=\tilde{\eta}_t$. Taking a partial derivative of (64)-(66) with respect to t, we have:

$$\begin{aligned}\tilde{\eta}_t &= \Sigma(x)\tilde{\eta}_x+G[\tilde{\psi},\tilde{\eta},\tilde{\theta},\dot{\tilde{\theta}}], & (76)\\ \tilde{\eta}_1(0,t) &= q\tilde{\eta}_2(0,t), & (77)\\ \tilde{\eta}_2(1,t) &= 0. & (78)\end{aligned}$$

Furthermore, taking a partial derivative of (76)-(78) with respect to t, we have:

$$\begin{aligned}\tilde{\vartheta}_t &= \Sigma(x)\tilde{\vartheta}_x+H[\tilde{\psi},\tilde{\eta},\tilde{\vartheta},\tilde{\theta},\dot{\tilde{\theta}},\ddot{\tilde{\theta}}], & (79)\\ \tilde{\vartheta}_1(0,t) &= q\tilde{\vartheta}_2(0,t), & (80)\\ \tilde{\vartheta}_2(1,t) &= 0. & (81)\end{aligned}$$

We introduce the following Lyapunov functionals:

$$\begin{aligned}V &= U+\frac{1}{2}\dot{\tilde{\theta}}(t)^2+\int_0^1\tilde{\eta}^{\intercal}(x,t)D(x,t)\tilde{\eta}(x,t)\,\mathrm{d}x, & (82)\\ W &= V+\frac{1}{2}\ddot{\tilde{\theta}}(t)^2+\int_0^1\tilde{\vartheta}^{\intercal}(x,t)D(x,t)\tilde{\vartheta}(x,t)\,\mathrm{d}x & (83)\end{aligned}$$

Applying the same steps like for U, for $\|\tilde{\psi}\|_\infty+\|\tilde{\eta}\|_\infty\leq\delta_2$, where $\delta_2>0$, we have:

$$\begin{aligned}\dot{V}(t) &\leq -\lambda_2V(t)+K_3U(t)^{3/2}+K_4\|\tilde{\psi}_x\|_\infty U(t)\\ &\quad +K_5V(t)^{3/2}+K_6\|\tilde{\eta}_x\|_\infty V(t), & (84)\\ \dot{W}(t) &\leq -\lambda_3W(t)+K_7U(t)^{3/2}+K_8\|\tilde{\psi}_x\|_\infty U(t)\\ &\quad +K_9V(t)^{3/2}+K_{10}\|\tilde{\eta}_x\|_\infty V(t)\\ &\quad +K_{11}\|\tilde{\psi}\|_\infty W(t)+K_{12}\|\tilde{\eta}\|_\infty W(t). & (85)\end{aligned}$$

The following inequalities are crucial in establishing the main result in this section.

$$\begin{aligned}\|\tilde{\vartheta}\|_\infty &\leq c_1\left(\|\tilde{\psi}_{xx}\|_\infty+\|\tilde{\psi}_x\|_\infty+\|\tilde{\psi}\|_\infty\right), & (86)\\ \|\tilde{\vartheta}\|_{\mathbb{L}^2} &\leq c_2\left(\|\tilde{\psi}_{xx}\|_{\mathbb{L}^2}+\|\tilde{\psi}_x\|_{\mathbb{L}^2}+\|\tilde{\psi}\|_{\mathbb{L}^2}\right), & (87)\\ \|\tilde{\psi}_{xx}\|_\infty &\leq c_3\left(\|\tilde{\vartheta}\|_\infty+\|\tilde{\eta}\|_\infty+\|\tilde{\psi}\|_\infty\right), & (88)\\ \|\tilde{\psi}_{xx}\|_{\mathbb{L}^2} &\leq c_4\left(\|\tilde{\vartheta}\|_{\mathbb{L}^2}+\|\tilde{\eta}\|_{\mathbb{L}^2}+\|\tilde{\psi}\|_{\mathbb{L}^2}\right). & (89)\end{aligned}$$

where c_1, c_2, c_3, and c_4 are positive constants. These inequalities can be obtained directly by bounding the norms for small $\tilde{\psi}$ in (64). Since $U(t)\leq V(t)\leq W(t)$, we have:

$$\dot{W}(t) \leq -\lambda_4W(t)+K_{13}W(t)^{3/2}. \quad (90)$$

Further, since $\|\tilde{\psi}\|_\infty+\|\tilde{\eta}\|_\infty\leq C_1W(t)$, then for sufficiently small $W(0)$, we have $W(t)\to 0$ exponentially. Since $W(t)$ is equivalent to the $\mathbb{H}^2$ norm of $\tilde{\psi}$ when $\|\tilde{\psi}\|_\infty+\|\tilde{\eta}\|_\infty$ is sufficiently small according to (86)-(89), there exists $\delta>0$ and c, such that if $\|\tilde{\psi}_0\|_{\mathbb{H}^2}\leq\delta$, then

$$\begin{aligned}&|\tilde{\theta}(t)|+\left|\dot{\tilde{\theta}}(t)\right|+\left|\ddot{\tilde{\theta}}(t)\right|+\|\tilde{\psi}\|^2_{\mathbb{H}^2}\leq\\ &ce^{-\lambda}\left(|\tilde{\theta}(0)|+\left|\dot{\tilde{\theta}}(0)\right|+\left|\ddot{\tilde{\theta}}(0)\right|+\|\tilde{\psi}_0\|^2_{\mathbb{H}^2}\right). \quad (91)\end{aligned}$$

Thus, we have the following result.

Theorem 2: Let $P(x,\xi)$ be the solution of (16)-(23). Then, there exists δ, such that for any $(\hat{u}_0,\hat{v}_0<\delta)\in\mathbb{H}^2(0,1)$, system (48)-(50) with $p(x)$ and $\hat{\theta}(t)$ are given by (27)-(28) and (31), has a unique classical solution $\hat{u}(x,t),\hat{v}(x,t)\in\mathbb{C}^{1,1}((0,1)\times(0,\infty))$. Furthermore, these estimates converge exponentially to the actual values u, v, and θ.

III. MANAGED PRESSURE DRILLING

A. Process Description

MPD is an advanced pressure control method that is used to precisely control the annular pressure throughout the wellbore in an oil well drilling. During drilling, a carefully designed fluid is pumped down from the mud pit through the drill string, through the drill bit, up the annulus around the drill string, and back to the mud pit. The aim is not only to transport cuttings in the annulus, but also to manage the pressure in the well so that the unwanted inflow from the surrounding formation or well fracturing can be avoided. The control system for MPD usually consists of two main components (**Fig**. 1):

- the hydraulics model that estimate the downhole pressure, and,

- a feedback control algorithm that automates the choke manifold to maintain the desired choke pressure.

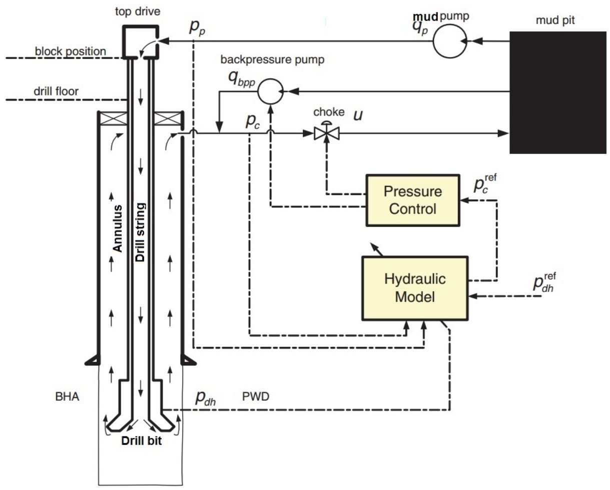

Fig. 1: Schematic of an automated MPD system (Statoil).

B. The Hydraulic Well Model

The hydraulic well model is used to estimate the downhole pressure and to provide the choke pressure set point for the MPD feedback control system. The main assumption is to consider the drilling fluid as a viscous fluid, so that the flow obeys the fundamental relations such as the equation of state, the mass conservation, the momentum conservation, and the energy conservation. The hydraulic well model is based on [8] and is outlined as follow. For a single-phase and one-dimensional flow in the annulus, the mass conservation is given by:

$$\rho_t(z,t) = -\frac{1}{A} m_z(z,t), \tag{92}$$

where ρ is the fluid density, A is the cross section area, m is the mass flow, t is the time instant, and $z \in [0, l]$ is the spatial coordinate along the flow path beginning from the downhole $z = 0$ to the topside $z = l$. The subscripts t and z denote partial derivatives with respect to t and z, respectively. Using the definition of the bulk modulus $\beta = \rho p_\rho$, (92) yields:

$$p_t(z,t) = -\frac{\beta}{A} q_z(z,t), \tag{93}$$

where q is the volumetric flow rate. The second relation can be obtained from momentum balance equation as follow:

$$\begin{aligned} m_t(z,t) &= -A p_z(z,t) - A\frac{\partial}{\partial z}\int_{\partial A} \rho v^2 \, \mathrm{d}A \\ &\quad -F_c(z,t) - A\rho g \sin\tau(z), \end{aligned} \tag{94}$$

where v is the fluid velocity and α is the angle between the positive flow direction and the horizon (for a vertical well $\alpha = 90$). F_c is the friction force acting on the volume. Assuming the integral term is sufficiently small, the flow rate equation is given by:

$$q_t(z,t) = -\frac{A}{\rho} p_z(z,t) - \frac{F_c(z,t)}{\rho} - Ag\sin\tau(z). \tag{95}$$

The boundary conditions are given by:

$$q(0,t) = q_p(t) + q_{lc}(t), \tag{96}$$

$$p(l,t) = p_c(t), \tag{97}$$

where q_p denotes the mud rate from the main pump, while q_{lc} denotes the volumetric rate of lost circulation which is an unknown parameter. The topside (choke) pressure is denoted by p_c. In this paper, the frictional pressure drop is modeled as follow:

$$F_c(z,t) = f_1 q(z,t) + f_2 q(z,t)^2. \tag{98}$$

where f_1 and f_2 denote the frictional coefficients.

C. Feasibility of the Design

The hydrostatic head can be removed from the momentum equation by defining:

$$\bar{p}(z,t) = p(z,t) - \rho g\left(l - \int_0^z \sin\tau(s)\,\mathrm{d}s\right). \tag{99}$$

The resulting system can be diagonalized using the following Riemann's coordinate transformation

$$\bar{u}(z,t) = \frac{1}{2}\left(q(z,t) + \frac{A}{\sqrt{\beta\rho}}\bar{p}(z,t)\right), \tag{100}$$

$$\bar{v}(z,t) = \frac{1}{2}\left(q(z,t) - \frac{A}{\sqrt{\beta\rho}}\bar{p}(z,t)\right). \tag{101}$$

Finally, defining $u(x,t) = \bar{u}(xl,t)$, $v(x,t) = \bar{v}(xl,t)$ and $w = [u \;\; v]^{\intercal}$, (93) and (95), together with the boundary conditions, can be written in a more compact form as follow:

$$w_t(x,t) = \Sigma w_x(x,t) + Cw(x,t) + f(w,x), \tag{102}$$

$$u(0,t) = -v(0,t) + q_{lc}(t), \tag{103}$$

$$v(1,t) = U(t), \tag{104}$$

where the unknown parameter $\theta = q_{lc}$, $\Sigma = \mathrm{diag}\left[-\frac{1}{l}\sqrt{\frac{\beta}{\rho}} \;\; \frac{1}{l}\sqrt{\frac{\beta}{\rho}}\right]$, and $C = \begin{pmatrix} 0 & -f_1/2\rho \\ -f_1/2\rho & 0 \end{pmatrix}$.

D. Explicit Observer Gains

The observer gain can be computed analytically in terms of Bessel function. Let $c = -\frac{F_1}{2\rho}$ and $\epsilon = \frac{1}{l}\sqrt{\frac{\beta}{\rho}}$, then the observer gain is the solution for the following system:

$$\epsilon P_x^{uu}(x,\xi) + \epsilon P_\xi^{uu}(x,\xi) = ce^{-2\frac{c}{\epsilon}x} P^{vu}(x,\xi), \tag{105}$$

$$\epsilon P_x^{vu}(x,\xi) - \epsilon P_\xi^{vu}(x,\xi) = -ce^{2\frac{c}{\epsilon}x} P^{uu}(x,\xi), \tag{106}$$

$$P^{uu}(0,\xi) = -P^{vu}(0,\xi), \tag{107}$$

$$P^{vu}(x,x) = -\frac{c}{2\epsilon} e^{2\frac{c}{\epsilon}x}. \tag{108}$$

Utilizing the results in [14], the solution is given by:

$$\begin{aligned} P^{vu}(x,\xi) &= -\frac{1}{2\epsilon}\left\{ cI_0\left[\frac{|c|}{\epsilon}\sqrt{\xi^2 - x^2}\right] \right. \\ &\quad \left. -|c|\sqrt{\frac{\xi - x}{\xi + x}} I_1\left[\frac{|c|}{\epsilon}\sqrt{\xi^2 - x^2}\right]\right\} \end{aligned} \tag{109}$$

$$\begin{aligned} P^{uu}(x,\xi) &= \frac{1}{2\epsilon}\left\{ cI_0\left[\frac{|c|}{\epsilon}\sqrt{\xi^2 - x^2}\right] \right. \\ &\quad \left. -|c|\sqrt{\frac{\xi + x}{\xi - x}} I_1\left[\frac{|c|}{\epsilon}\sqrt{\xi^2 - x^2}\right]\right\} \end{aligned} \tag{110}$$

where I_n is the modified Bessel function of the first kind of order n.

IV. EXPERIMENT DESIGN

The experiment is carried out in a field scale flow-loop test in Stavanger, Norway by Statoil. The MPD system is modeled as a U-tube (**Fig**. 2) which consists of the main pump, 700 meters of pipes (each for drill string and annulus), the downhole assembly, and the topside sensors. The lost circulation happen when drilling fluid flows into geological formations instead of returning up the annulus. The downhole assembly is completed with an exit valve to simulate the lost circulation problem and a Coriolis meter to measure the lost circulation rate. The topside sensor consists of a Coriolis meter to measure the return fluid and a pressure gauge to measure the pressure.

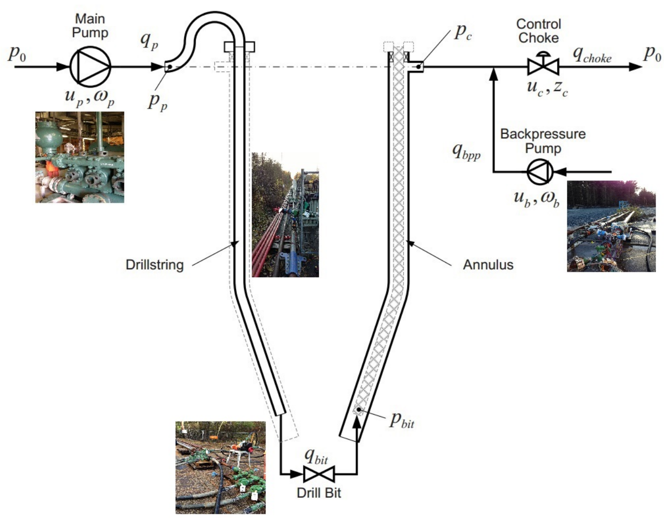

Fig. 2: Schematics of the flow-loop.

In this experiment, water is injected by the main pump through the drill string and up the annulus. After some time, the exit valve is gradually opened to simulate the lost circulation. The task is to estimate the downhole pressure along the annulus and the lost circulation rate using only measurements at the topside of the well.

The measured volumetric water flow and pressure from the Coriolis meter and the pressure gauge at the topside of the well are presented in **Fig**. 3. In practice, this is the only reliable measurement during drilling. The Coriolis meter gives more noise than does the pressure gauge. This noise is inherent to the estimations. To reduce the noises, the volumetric flow rate is filtered using a robust method of local regression with weighted linear squares and 2nd-degree polynomial model. **Fig**. 4 show the relation of the volumetric flow rate of lost circulation and the valve opening. It can be observed that the valve opening is almost linear with the flow rate.

V. RESULTS AND DISCUSSIONS

Relying only on the topside measurements, we want to estimate the downhole pressure and the rate of lost circulation.

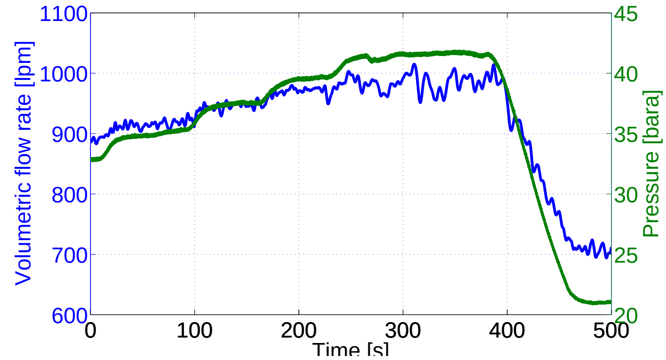

Fig. 3: Measured topside flow rate (coriolis meter) and pressure.

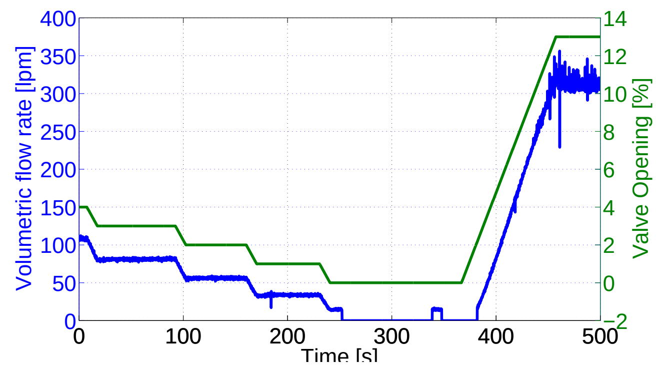

Fig. 4: Downhole exit valve opening and volumetric flow rate response.

Using our adaptive observer (48)-(50), where an unknown parameter is estimated using the following update-law:

$$\hat{q}_{lc} = \frac{\kappa}{2}\left(q_c(t) - \hat{q}_c(t) + \frac{A}{\sqrt{\beta\rho}}(p_c(t) - \hat{p}_c(t))\right), \quad (111)$$

where q_c and p_c are measurements taken at the topside of the well. The estimated and measured downhole pressure and the rate of lost circulation can be seen from **Fig**. 5 and **Fig**. 6, respectively. No control is applied in this experiment. The update-law gain κ is computed with a simple line search algorithm. The estimated and measured topside pressure are in good agreement, as shown in **Fig**. 7.

VI. CONCLUSIONS AND FUTURE WORKS

State and parameter estimations for the nonlinear hyperbolic PDEs have been presented in this paper. The proposed designs, which are based on the backstepping method, have been successfully implemented and the result has been validated with a full-scale flow-loop test conducted in Stavanger by Statoil. The design for the nonlinear system works locally and for sufficiently small initial values. Because only one phase (fluid) is involved, the annular model is represented by a one-phase incompressible flow model. There is an interest to include the gas phase when drilling a well below its formation pressure, leading to a two-phase flow model (refers to the drift flux model). This will be the topic of future works.

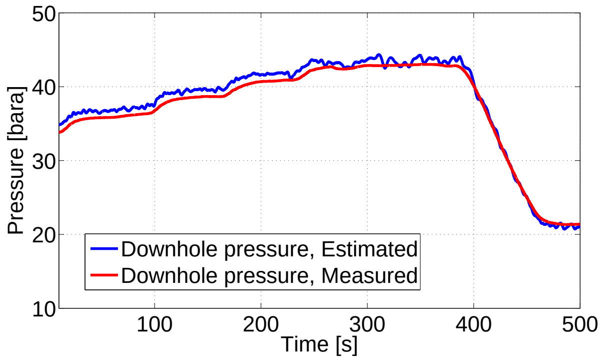

Fig. 5: Estimated and measured downhole pressure.

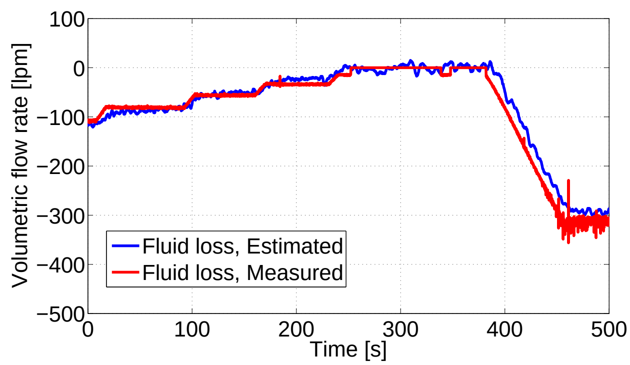

Fig. 6: Estimated and measured fluid loss at the bottom of the well.

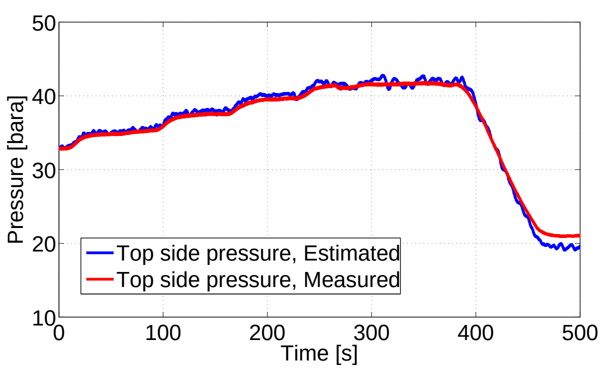

Fig. 7: Estimated and measured topside pressure.

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