TESTING THE POWER LAW ON URBAN WATER AND WASTEWATER PIPELINE NETWORKS

Ev norsk rubrik

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Abstract

Active research has been going on to observe and validate the Power Law in physics, computer science, economics, linguistics, sociology, geophysics etc. This paper sets out to test the hypothesis that the Power Law is a feature of water and wastewater pipeline networks in cities. Databases from 30 different municipalities in Norway serve as the raw data to be processed. The findings may seem a bit intuitive in retrospect, but the numerical results provide some interesting insights. There is greater conformity to the Power Law when the number of pipelines in different length-classes is taken into account, as compared to when the total length in each length-class is considered. By considering only the equations with a high degree of conformity to the Power Law, it is possible to derive a standard equation representing an ‘average saturated Norwegian water and wastewater pipeline network.’ The authors recommend similar analyses of networks in other parts of the world in order to confirm the possible existence of the Golden Ratio (the solution of the equation $x^2 - x - 1 = 0$) and the Euler number (the base of the natural logarithm) e, in the scheme of things.

Key words – Urban water and wastewater pipeline networks, water pipelines, wastewater pipelines, power law, pipeline stock analysis

Sammanfattning

Forskning har forsøkt å observere og validere gyldigheten til ”Power Law” innenfor fysikk, informatikk, økonomi, språkvitenskap, sosiologi, geofysikk osv. Denne artikkelen forsøker å teste hypotesen om at ”Power Law” kan knyttes til ledningsnettverk for vann- og avløpsystemer i byer, hvilket vil innebære en grad av isomorfisme med andre tidligere nevnte felt. Undersøkelser er først foretatt for ulike rørledningslengder i ulike størrelsesklasser, så vel som for totallengder av disse. Databaser fra 30 ulike kommuner i Norge representerer datagrunnlaget i undersøkelsen. Funnet kan i etterkant synes intuitivt, men de numeriske resultatene gir ny og interessant innsikt. Når antall rørledninger i ulike lengdeklasser tas i betraktning er det stor overenstemmelse med ”Power Law”, sammenlignet med tilfellet for total lengde innenfor hver lengdeklasse. Ved kun å inkludere de likninger som har stort samsvar med ”Power Law” er det mulig å utlede en standardformel som representerer et ’gjennomsnittlig mettet ledningsnettverk for norske vann- og avløpsystemer’. Forfatterne anbefaler lignende analyser av nettverk i andre deler av verden for å bekrefte den mulige eksistensen av et Gyllent Snitt (lösningen av likningen $x^2 - x - 1 = 0$) og Eulers tall (basen for den naturlige logaritmen) e, i verdensordenen.

1 Introduction

The urban infrastructure in the Western world, and its asset management, faces several challenges. First of all, it is a well-known fact that much of the infrastructure is ageing, owing to the fact that it was constructed many decades ago. Secondly, the rehabilitation rate of infrastructure systems generally suffers from a severe backlog problem, where annual rehabilitation and replacement investments do not keep up with the ageing process.
Hence, both the functional quality and the asset value of the infrastructure deteriorate over time. Thirdly, the urban population in many industrialised countries tends to stagnate, and as a result, the infrastructure stocks saturate – the stocks do not increase at the same rate as it used to in the past. Rather, the stocks stop growing and thereby diminish when measured on the basis of per-capita-population-serviced.

In such a situation, utility owners need to emphasise more on the operation, maintenance and management issues, rather than new investments. Therefore, it becomes all the more important to understand better the critical stock characteristics of built environment, such as stock state parameters (size, composition and age), stock dynamic parameters (rate of new construction, rehabilitation and demolition), and stock performance parameters (resource consumption, energy inputs, waste flows, costs and environmental impacts), cfr. Kohler and Yang (2007).

This paper examines the urban water cycle pipeline networks of 30 cities/towns in Norway, i.e., the water supply and the stormwater and sewage collection pipeline networks of the urban areas that we consider to be representative for the present urban water cycle infrastructure in this country. We test to what extent the pipeline network stock, as of today, can be characterised by the so-called “Power Law”. We examine two state parameters for this kind of stock – how the number of pipelines is distributed among given pipeline length classes, and how the length of pipelines is distributed among the same pipeline length classes. Both these types of parameters can be referred to as stock state parameters, according to Kohler and Yang (2007). If the Power Law is valid for urban pipeline systems, one may estimate these two important state parameters in a more easy and consistent way, mathematically, than by doing a time-consuming case-based empirical stock accounting exercise. To our knowledge, this work is original and the first attempt internationally to test the Power Law on urban water cycle infrastructure.

### 1.1 Pipeline networks

Urban water and wastewater pipeline networks are key components of urban infrastructure assets. They are put in place to serve two primary requirements – the supply and distribution of water for consumption, and the transport of sewage and stormwater to wastewater treatment plants and recipients. Within a given pipeline network, one finds pipelines of varying diameters, materials of construction, lengths and thicknesses. As far as the sizes (diameters) are concerned, a network is categorised basically into large-size, medium-diameter and small-size pipe-classes. The minima and maxima of the ranges for each of these three classes vary from network to network. Within each of these size classes, one finds pipelines of different lengths. Pipeline networks also evolve over time. The evolution is associated with material mass inflows into the network, and disconnections of pipelines from the network (outflows). The stock of (active plus inactive) pipelines at any instant of time is a consequence of material flows into the network that or point in time.

Numerous studies have been carried out on various aspects of water and wastewater pipeline networks – stocks and flows, life-cycle energy consumption, operation, maintenance and rehabilitation expenses, environmental impacts, rehabilitation strategies etc. Venkatesh et al. (2009) and Venkatesh et al. (2010) have based environmental life cycle assessment of the wastewater and water pipeline networks in Oslo, respectively, on a foreknowledge of the mass (and energy) flows. Ugarelli et al. (2008) has presented a stock analysis of the wastewater pipeline network in Oslo, as a background to investigating the superiority of a physical lifetime approach to rehabilitation of pipelines, as compared to an economic lifetime approach. Venkatesh et al. (2010) has performed a study of pipeline material masses in the wastewater and water pipeline networks in Oslo, Trondheim and Tromsø, and found that there is a possibility of a correlation between the population density and the per-capita-active-pipeline-material-masses-in-stock, which needs to be verified and confirmed by considering more datasets. An optimum asset management strategy is based on knowing the asset better. This would necessitate the maintenance of an information system (database) that would track assets and keep a tab on costs and reliability, as pointed out by ASCE (1999). Such an information system which records location, condition and criticality of assets (more relevant in the case of pipeline networks) enables effective asset management (WERF, 2004). In a saturated pipeline network – one in which no significant pipeline additions are being made – rehabilitation and repair dominate asset management. Sægrov (2004; 2005; 2006) presented tools which can be utilised by water and wastewater utilities to manage their rehabilitation activities. Ambrose et al. (2005) have carried out an embodied energy analysis for water and wastewater pipelines in Australia in order to determine the energy that is expended throughout the life-cycle of pipelines in the network.

In addition to understanding the material composition, the age-type (water, stormwater, sewage or combined flow) distribution and geographical locations of pipelines in a pipeline stock analysis, analysing the composition in terms of the lengths of the individual pipelines would add to a more complete understanding of the networks. The authors did not come across any such
study in literature and thereby would like to posit this analysis as an important and original contribution. This paper attempts to do this by testing the hypothesis that the Power Law is applicable to urban water and wastewater pipeline networks. The cases considered in the paper (depending solely on accessibility for the authors) are networks as at the end of year 2009 for a host of Norwegian municipalities. The networks are of different sizes, but we expect that they would all be fairly close to a state of saturation, and the rate of growth in all the networks, in the years to come, can be considered to be very small. The methodology is described in the next section, followed by the results of the tests and discussions thereof.

2 Theory

2.1 The Power Law

Newton’s Law of Gravitation, the Coulomb force equation, Gutenberg-Richter Law for earthquake sizes, Pareto’s Law of income distribution (the famed ‘80-20 Law’), Horton’s Law of river systems, Richardson’s Law of severity of violent conflicts, and even Bradford’s Law of citations of journal papers, are all classic examples of the Power Law, which can be expressed generally by the following power equation:

\[ y = a \cdot x^k \]  

(1)

where \( y \) and \( x \) are the variables related by the Power Law, and \( a \) and \( k \) are constants. Taking the logarithms on both sides of equation (1) yields equation (2), which is that of a straight line between \( \log(y) \) and \( \log(x) \), having a slope of \( k \), and intersecting the Y-axis at the point \( \log(a) \).

\[ \log(y) = \log(a) + k \cdot \log(x) \]  

(2)

We can write equation (2) in a simplified way:

\[ Y = A + k \cdot X \]  

(3)

where \( Y \) represents \( \log(y) \), \( A \) represents \( \log(a) \), and \( X \) represents \( \log(x) \).

As an elucidation, one could consider the simple equation for the volume of a cube; \( V = L^3 \), where \( V \) is the volume and \( L \) is the length of the side. This means that there is a Power Law relationship between the volume of the cube and its side – a power of three in this case. Research has validated the Power Law type of distribution in the frequency of words in a text, population of cities, the GDP per capita of countries (Gulimi et al, 2003), the hyperlinks from/to websites on the Worldwide Web (Shiode and Batty, 2010), \textit{inter alia}. Philip Ball, in his award-winning best-seller Critical Mass (published in 2004 in the UK), has dwelt on the relevance of the Power Law to manmade and natural systems. There is the phenomenon of isomorphism in General Systems Theory which states that there are common features or properties in systems even if the systems are of rather different natures. The Power Law is one such common feature.

3 Background and Methodology

Requests for data were sent out by e-mail to all 431 municipalities in Norway in the period 2008–2010. Each municipality administers a water and wastewater pipeline network catering to the needs of the residents within its domain. All active, operating pipelines – water, sewage, combined flow and stormwater – for each city/town are considered together as constituents of an aggregated pipeline network. The individuals at the different municipalities, who responded and provided the authors with the databases sought, are identified in the List of Respondents, in the end of this paper. Data amenable to the analysis were received from 30 municipalities, including the three largest cities in the country – Oslo, Bergen and Trondheim – which together account for 23 % of the total population. While the number of municipalities considered in the analysis may be just 7 % of the total, the serviced-population accounted for, is over 1.63 million – well over 35 % of the total. On a metres per capita basis, the maximum is for the municipality of Aseral (223.3), and the minimum for Oslo (6.11). The municipalities, whose locations have been indicated on the map in Figure 1 by the black dots, are listed in Table 1 in alphabetical order. The population statistics for the year 2010 are sourced from Statistics Norway.

The tests are carried out first by considering the number of pipelines in each length-category. The class width is set at 100 metres. The variable \( x \) thus is the midpoint of each class width (50, 150, 250 and so on). The variable \( y \) is the number-fraction for each class – in other words, the probability of finding a pipeline with length belonging to that class. We then represent \( \log(x) \) by \( X \), \( \log(y) \) by \( Y \) and \( \log(a) \) by \( A \), as in equation (3). Values of \( X \) and \( Y \) are then regressed in order to obtain the best-fit lines and the equations thereof, for each municipality, see Table 1, including the correlation coefficient, \( R^2 \). Hereafter in the text, these equations are referred to as ‘number-equations.’ The tests are then repeated by considering the total length of all pipelines in each length-category for the same class width of 100 metres; and the ‘length-equations’ are similarly obtained.

The number of equations is then whittled down by
setting cut-off $R^2$ values for each case. Average best-fit lines are then constructed in order to represent roughly any saturated urban water and wastewater pipeline network in Norway. The nature and the implications of the equations are subsequently discussed. Further work, as a continuation of this paper, is recommended towards the end.

4 Results and discussions

4.1 Number of pipes

Referring to Table 1, for a class-width of 100 metres, it is seen that the $R^2$ value for all the municipalities except Åseral is greater than 0.85, indicating, in general an appreciable conformity to the Power Law. The arithmetic average $R^2$ value for all the municipalities is a healthy 0.94 (with a low standard deviation of 0.04). The $k$ value varies from a low of –1.5 for Åseral, which emerges as a distinct outlier in this respect as well, to a high of –3.4 for the city of Bergen. The arithmetic average value of $k$ turns out to be –2.54 (standard deviation of 0.4). While in general, one may, on the basis of the equations obtained for the said municipalities conclude that the Power Law is applicable to the number of pipelines in the urban water and wastewater pipeline networks, one could also identify different ‘degrees of conformity’ to the Law. A higher degree of obedience would translate to a higher $R^2$ value. If 0.95 is assumed to be a cut-off, and all $k$ corresponding to $R^2$ values greater than 0.95 are considered, one is left with 16 values (of the total of 30). The average of these 16 $k$ values is –2.69 (standard deviation of 0.32). The authors admit that data sometimes can be tortured to yield pre-determined conclusions. However, while setting out to establish whether the Power Law is applicable to water and wastewater networks, the authors may possibly have serendipitously arrived at a possible constant $k$ which converges towards $-e$ (negative of the Euler number which is used as the base of the Napierian logarithm – 2.718).

Further, the 16 equations with $R^2$ values greater than 0.95 are culled out and the corresponding best-fit lines are plotted together in Figure 3. The averages, maxima and minima of the log(y) values for each log(x) are determined. When the y values corresponding to the averages of the log(y) values are summed up, the total is 0.992. In other words, the sum of the probabilities is equal to 0.992 (very close to 1, which it should ideally be). If a best-fit line is plotted for the points represented by log(x) and the corresponding average of the log(y) values (the thick black line in Figure 2), the equation is:

$$Y = 4.42 - 2.63 \cdot X$$  

(4)

The $R^2$ value of the line represented by equation (4) is a satisfactory 0.939. The dotted lines on either side of the average line indicate the 95% confidence intervals and move closer to the average line as the value of log(x) increases. This equation, which can be looked upon as an approximate description of a saturated water and wastewater pipeline network in Norway, it must be recalled, is class-width-specific. In other words, any value of x which is chosen is the mid-point of a class which is 100 m wide: 50 m on either side of the x value. This means that the least value of x which can be chosen is 50. Figure 3 plots the results of a dry run of equation (4). It is seen from Figure 3 that 56.8% of all pipelines in a network will have their lengths between 10 and 110 m (with x = 60 m as the midpoint of the class-width), 14.8% will have their lengths between 50 and 150 m (with x = 100 m), and so on. As the length increases, the probability of finding a pipe of that length in the network decreases. For instance, the probability of finding a pipeline with its length in the range 450 to 500 m, is a measly 0.22% – which means that only one out of 455 pipelines statistically would have its length in the said range. This equation is likely to hold, within tolerable error limits, for most saturated water-wastewater pipeline networks in Norway, conforming appreciably to the Power Law.

4.2 Length of pipes

Referring again to Table 1 and focusing on the $k$ and $R^2$ values for the equations describing the lengths of pipelines in the networks, it is seen that there is lesser conformity to the Power Law vis-à-vis the number-equations. The average of the $R^2$ values is 0.83, with a standard deviation of 0.14. If 0.85 is set as a cut-off point, one ends up with 18 equations (60% of the total), with an average $R^2$ of over 0.902. The average $k$ value for all the 30 equations is –1.41 and it increases to –1.602 for $R^2$ values for the 18 equations referred to. Quite similar to the possibility of the existence of the Euler number in the previous case, the authors recommend further investigation to confirm the existence of the so-called Golden Ratio (1.618; the positive solution of the equation $x^2 - x - 1 = 0$) in the case of the length-equations.

The exercises carried out for Figure 2 and Figure 3 are repeated for this case as well, and illustrated in Figures 4 and 5. The average line (thick black line in Figure 4) has the equation:

$$Y = 2.51 - 1.613X$$  

(5)

The $R^2$ value of the line represented by Equation 5 is 0.87; indicating a good conformity, though less than that of Equation 4. The dotted lines on either side of the
average of the average values of log(y) – the sum of the probabilities in other words – is 0.96 (or 96%); whereas it should ideally be equal to 1. It is seen that as the $R^2$ value increases from an average of 0.83 to 0.87 to 0.902, the $k$ value also increases from $-1.41$ to $-1.466$ to $-1.602$; a possible indication if one may say so, of a convergence to the Golden Ratio, as the conformity to the Power Law increases. Equation (5) could be considered to be typically representative of most saturated water and wastewater pipeline networks in Norway, conforming appreciably to the Power Law.

The values in Figure 5 can be interpreted thus. It is seen that 44.4% of all pipelines in a network will have their lengths between 10 and 110 metres (with $x = 60$ m as their average), 19.5% will have their lengths between 50 and 150 m (with $x = 100$ m), and so on. As the length increases, the probability of finding a pipe of that length in the network decreases. For instance, the probability of finding a pipeline with its length in the range 450 to 500 m, is a measly 0.22% – which means that only one out of 455 pipelines statistically would have its length in the said range. If Figure 3 is compared with Figure 5, it is noted that for the 100-metre wide classes with mid-point values 60, 75, 80 and 90 m, the contribution to the total number of pipelines in the network is greater than the share in the total length of the pipelines. The value of $x$ which equates the $y$ in equation (4) to that in equation (5) is 121 m. In other words, the 100-m wide class ranging from 71 to 171 m has a contribution to the total number of pipelines which is the same as its contribution to the total length of pipelines (15.1%). For all the 100-m wide classes with the minima greater than 71 m, the contribution to the total length is greater than that to the total number of pipelines.

6 Conclusions and recommendations

This paper based its analysis on databases obtained from municipalities in Norway. While all 431 municipalities in the country (as in year 2010) were contacted, databases amenable to the analysis were obtained from 30 of them. Whether 30 databases (and thereby 30 equations) are sufficient to draw conclusions and generalisations is debatable, though the authors would like to believe that it is also certainly not too small so as to deter such an analysis.

When the number of pipelines in each pipe-length class was taken into consideration, a much greater conformity to the Power Law was observed, vis-à-vis the lengths of the pipelines in each class. The reason for this has been discussed in the previous section. By segregating the equations on the basis of degree of conformity and setting cut-off points for the $R^2$ values, the number of equations was whittled down in each case. The average best-fit line was then determined for each case, and it was found out that the $R^2$ values were satisfactory enough.

By intuitive reasoning, one could aver that there would be a greater number of shorter pipes in the network as compared to longer ones (and thereby a higher componentry). However, it is certainly insightful that pipeline networks tend to obey the Power Law – a little more so when the number of pipelines in each size category is considered. It may not be right to make any specific comments about the exponential constant $k$ (as in equation (1)) for the networks in Norway, even though the authors wonder if it could be true that the Golden Ratio and the Euler’s number are defining features of pipeline networks (in Norway and in general). What this paper does thereby is to leave some food for thought for readers, and stimulate further thinking in this direction.

This exercise can be tried out for pipeline networks in even more towns and cities. Networks in different parts of the world can also be compared with each other in this regard. Networks keep growing when cities expand, and the equations of the best-fit Power Law equations describing them would keep changing. It is also proba-
ble that as a pipeline network moves towards saturation, its conformity to the Power Law would improve. What is important is the fact that irrespective of the size of the pipeline network (total number of pipes and total length of all the pipelines taken together), not only does the Power Law hold good, but the exponent $k$ in equation (1) seems to converge towards known mathematical constants. However, as referred to in the previous paragraph, this is just a happenstance (possibly serendipitous) finding which needs to be tested rigorously in future works of this nature.

The authors, as a furtherance of this study, are also working on investigating the existence of a possible correlation (and if there exists one, the nature of the same) between the per-capita mass of pipeline materials of construction in the active water/wastewater pipeline network and the population density. In a preliminary investigation (Venkatesh et al., 2010) with the datasets of only three cities – Trondheim, Oslo and Tromsø – the authors ended up with an ambivalent result. There is a clear correlation – supported with very high $R^2$ values – but the nature of the correlation is uncertain, as the $R^2$ values are high enough (above 0.94) for logarithmic, linear, quadratic polynomial and Power Law relationships. Testing with more datasets will be the next step in resolving this uncertainty. That will also add another dimension to stocks analysis, with regard to its functionality (the population served).

In conclusion, one can state that this is another step forward in substantiating what Ludwig von Bertalanffy said in his work on General Systems Theory in 1968 – There exist models and laws that apply to generalised systems irrespective of their particular kind. It seems logical to look for universal principles applying to systems in general. The Power Law could similarly be tested for several other aspects of the built environment, in order to improve the understanding of anthropogenic assets, for improved asset management necessitates knowing systems better (WERF, 2004).

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References


Table 1. Regression results for pipeline network characterisation (using class-width of 100 m).

<table>
<thead>
<tr>
<th>City/town</th>
<th>Population served (in 2010) (capita)</th>
<th>Total length of pipelines (metres)</th>
<th>Specific length of pipelines (m/cap)</th>
<th>Regressions based on number of pipes (Y = A + k·X)</th>
<th>( R^2 ) value</th>
<th>Regressions based on length of pipes (Y = A + k·X)</th>
<th>( R^2 ) value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ålesund</td>
<td>42982</td>
<td>725248</td>
<td>16.9</td>
<td>( Y = 4.81–2.86X )</td>
<td>0.950</td>
<td>( Y = 2.87–1.75X )</td>
<td>0.880</td>
</tr>
<tr>
<td>Alta</td>
<td>18680</td>
<td>390834</td>
<td>20.9</td>
<td>( Y = 4.67–2.63X )</td>
<td>0.940</td>
<td>( Y = 2.68–1.59X )</td>
<td>0.843</td>
</tr>
<tr>
<td>Åseral</td>
<td>917</td>
<td>204795</td>
<td>223.3</td>
<td>( Y = 1.86–1.50X )</td>
<td>0.781</td>
<td>( Y = 0.28–0.43X )</td>
<td>0.238</td>
</tr>
<tr>
<td>Bergen</td>
<td>256600</td>
<td>3056438</td>
<td>12.0</td>
<td>( Y = 5.83–3.42X )</td>
<td>0.970</td>
<td>( Y = 3.89–2.25X )</td>
<td>0.937</td>
</tr>
<tr>
<td>Bodø</td>
<td>47282</td>
<td>1206235</td>
<td>25.5</td>
<td>( Y = 3.41–2.33X )</td>
<td>0.919</td>
<td>( Y = 1.58–1.23X )</td>
<td>0.790</td>
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<tr>
<td>Eigersund</td>
<td>14170</td>
<td>198741</td>
<td>14.0</td>
<td>( Y = 3.82–2.31X )</td>
<td>0.974</td>
<td>( Y = 1.88–1.27X )</td>
<td>0.928</td>
</tr>
<tr>
<td>Fet</td>
<td>10238</td>
<td>262000</td>
<td>25.6</td>
<td>( Y = 4.84–2.76X )</td>
<td>0.940</td>
<td>( Y = 2.87–1.71X )</td>
<td>0.853</td>
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<tr>
<td>Gran</td>
<td>13363</td>
<td>299003</td>
<td>22.4</td>
<td>( Y = 4.58–2.59X )</td>
<td>0.941</td>
<td>( Y = 2.55–1.54X )</td>
<td>0.850</td>
</tr>
<tr>
<td>Hamar</td>
<td>28344</td>
<td>739955</td>
<td>26.1</td>
<td>( Y = 5.02–2.92X )</td>
<td>0.970</td>
<td>( Y = 3.14–1.87X )</td>
<td>0.930</td>
</tr>
<tr>
<td>Kvar</td>
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<td>464088</td>
<td>55.5</td>
<td>( Y = 4.17–2.49X )</td>
<td>0.950</td>
<td>( Y = 2.15–1.40X )</td>
<td>0.850</td>
</tr>
<tr>
<td>Larvik</td>
<td>42412</td>
<td>1164706</td>
<td>27.5</td>
<td>( Y = 4.42–2.62X )</td>
<td>0.962</td>
<td>( Y = 2.51–1.55X )</td>
<td>0.895</td>
</tr>
<tr>
<td>Marker</td>
<td>3471</td>
<td>54846</td>
<td>15.8</td>
<td>( Y = 3.12–1.93X )</td>
<td>0.953</td>
<td>( Y = 1.05–0.86X )</td>
<td>0.820</td>
</tr>
<tr>
<td>Moss</td>
<td>30030</td>
<td>375806</td>
<td>12.5</td>
<td>( Y = 4.63–2.79X )</td>
<td>0.967</td>
<td>( Y = 1.81–1.39X )</td>
<td>0.741</td>
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<tr>
<td>Nord-Odal</td>
<td>5118</td>
<td>119302</td>
<td>23.3</td>
<td>( Y = 4.21–2.45X )</td>
<td>0.972</td>
<td>( Y = 2.22–1.39X )</td>
<td>0.913</td>
</tr>
<tr>
<td>Odda</td>
<td>7047</td>
<td>215190</td>
<td>30.5</td>
<td>( Y = 5.12–2.93X )</td>
<td>0.941</td>
<td>( Y = 3.11–1.84X )</td>
<td>0.850</td>
</tr>
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<td>181914</td>
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<td>( Y = 4.33–2.57X )</td>
<td>0.986</td>
<td>( Y = 2.44–1.53X )</td>
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<tr>
<td>Orkdal</td>
<td>11276</td>
<td>395118</td>
<td>35.0</td>
<td>( Y = 3.47–2.19X )</td>
<td>0.930</td>
<td>( Y = 1.52–1.14X )</td>
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<tr>
<td>Oslo</td>
<td>586860</td>
<td>3591000</td>
<td>6.1</td>
<td>( Y = 3.98–2.74X )</td>
<td>0.896</td>
<td>( Y = 2.14–1.64X )</td>
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<td>Rennesøy</td>
<td>4035</td>
<td>82578</td>
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<td>( Y = 3.07–1.86X )</td>
<td>0.930</td>
<td>( Y = 0.93–0.82X )</td>
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<tr>
<td>Ringerikke</td>
<td>28806</td>
<td>761550</td>
<td>26.4</td>
<td>( Y = 5.16–2.94X )</td>
<td>0.970</td>
<td>( Y = 3.19–1.84X )</td>
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<tr>
<td>Sandefjord</td>
<td>43126</td>
<td>761450</td>
<td>17.7</td>
<td>( Y = 5.12–2.92X )</td>
<td>0.961</td>
<td>( Y = 2.93–1.77X )</td>
<td>0.890</td>
</tr>
<tr>
<td>Sandnes</td>
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<td>1817213</td>
<td>28.1</td>
<td>( Y = 3.26–2.43X )</td>
<td>0.885</td>
<td>( Y = 1.50–1.29X )</td>
<td>0.730</td>
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<tr>
<td>Steinkjer</td>
<td>21050</td>
<td>985202</td>
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<td>0.860</td>
<td>( Y = 0.85–0.88X )</td>
<td>0.580</td>
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<tr>
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<td>21375</td>
<td>680054</td>
<td>31.8</td>
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<td>0.984</td>
<td>( Y = 1.73–0.62X )</td>
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<td>( Y = 0.88–0.88X )</td>
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<tr>
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<td>23.0</td>
<td>( Y = 4.50–2.60X )</td>
<td>0.973</td>
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<td>67305</td>
<td>792000</td>
<td>11.8</td>
<td>( Y = 4.14–2.50X )</td>
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<td>( Y = 2.39–1.56X )</td>
<td>0.862</td>
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<tr>
<td>Trondheim</td>
<td>170936</td>
<td>1843000</td>
<td>10.8</td>
<td>( Y = 4.51–2.75X )</td>
<td>0.903</td>
<td>( Y = 2.54–1.66X )</td>
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<tr>
<td>Tønsberg</td>
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<td>534365</td>
<td>13.6</td>
<td>( Y = 5.10–2.87X )</td>
<td>0.975</td>
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</tr>
<tr>
<td>Østre Toten</td>
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<td>817503</td>
<td>56.3</td>
<td>( Y = 4.78–2.69X )</td>
<td>0.977</td>
<td>( Y = 2.74–1.61X )</td>
<td>0.923</td>
</tr>
</tbody>
</table>
Figure 1. Municipalities considered for the analysis.

Figure 2. Best-fit average of regression lines with $R^2$ values greater than 0.95 and its 95% confidence intervals.

Figure 3. Dry-run of Equation 4.
Figure 4. Best-fit average of regression lines with $R^2$ values greater than 0.85 and its 95% confidence intervals.

Figure 5. Dry-run of Equation 5.