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Discussion paper

Recursive utility using the stochastic maximum principle

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Abstract

Motivated by the problems of the conventional model in rationalizing market data, we derive the equilibrium interest rate and risk premiums using recursive utility in a continuous time model. We consider the version of recursive utility which gives the most unambiguous separation of risk preference from time substitution, and use the stochastic maximum principle to analyze the model. This method uses forward/backward stochastic differential equations. With existence granted, the market portfolio is determined in terms of future utility and aggregate consumption in equilibrium. The equilibrium real interest rate is also derived, and the the model is shown to be consistent with reasonable values of the parameters of the utility function when calibrated to market data, under various assumptions.

KEYWORDS: The equity premium puzzle, the risk-free rate puzzle, recursive utility, the stochastic maximum principle.

JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Questions like the following are sometimes asked: Are asset prices too volatile relative to the information

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arriving in the market? Is the mean risk premium on equities over the riskless rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

Mehra and Prescott (1985) raised some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889 - 1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations ¹.

We reconsider recursive utility in continuous time along the lines of Duffie and Epstein (1992a-b). In the first paper two versions of recursive utility were established, where one version was analyzed by the use of dynamic programming. The version left out is the one that gives the most unambiguous separation of risk preference from time substitution, which is the one we analyze in this paper. In doing so we use the stochastic maximum principle. The resulting model we solve, and present both risk premiums and the equilibrium interest rate. The method we use allows the volatilities in the model to be both time and state dependent.

Aside from this relaxation of the standard assumptions, we use the basic framework developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994), which elaborate the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models. The data set we use to calibrate the model is the same as the one used by Mehra and Prescott (1985) in their seminal paper on this subject. These suggest that the volatilities mentioned above are not the same.

¹Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

We also analyze the version treated in Duffie and Epstein (1992a) using our method, and obtain the same risk premiums. In addition we have an expression for the real interest rate.

Generally one can not assumed that all income is investment income. We assume that one can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. In reality the latter is not traded, so the return to the wealth portfolio is not readily observable or estimable from available data. We indicate how the model may be slightly adjusted under various assumptions, when the market portfolio is not a proxy for the wealth portfolio.

Besides giving new insights about these interconnected puzzles, the recursive model is likely to lead to many other results that are difficult, or impossible, to obtain using, for example, the conventional, time additive Eu-model. One example included in this paper is that we can explain the empirical regularities for Government bills.²

It has been a goal in the modern theory of asset pricing to internalize probability distributions of financial assets. To a large extent this has been achieved in our approach. Consider the logic of this Lucas-style model. Aggregate consumption is a given diffusion process. The solution of a system of forward/backward stochastic differential equations (FBSDE) provide the main characteristics in the probability distributions of future utility. With existence of a solution to the FBSDE granted, market clearing finally determines the characteristics in the market portfolio from the corresponding characteristics of the utility and aggregate consumption processes.

The paper is organized as follows: Section 2 starts with a brief introduction to recursive utility in continuous time, in Section 3 we derive the first order conditions, Section 4 details the financial market, in Section 5 we analyze the main version of recursive utility. In Section 6 we summarize the main results, and present some calibrations. Section 7 explores various alternatives when the market portfolio is not a proxy for the wealth portfolio, Section 8 points out some extensions, and Section 9 concludes. In the Appendix we present a derivation for the ordinally equivalent version, using

²There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen, Heaton, Lee (2008), Hansen and Scheinkman (2009), Wachter (2012), Bansal and Yaron (2004), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990 b), and Ai (2012) to name some important contributions. Related work is also in Browning et. al. (1999), and on consumption see Attanasio (1999). Bansal and Yaron (2004) study a richer economic environment than we employ.

the stochastic maximum principle.

2 Recursive Stochastic Differentiable Utility

In this section we recall the essentials of recursive, stochastic, differentiable utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994).

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0, T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Brownian motion driven uncertainty, N risky securities and one riskless asset (Section 5 provides more details). Consumption processes are chosen from the space L of square integrable progressively measurable processes with values in \mathbb{R}_+ .

The stochastic differential utility $U : L \rightarrow \mathbb{R}$ is defined as follows by two primitive functions: $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $A : \mathbb{R} \rightarrow \mathbb{R}$, where \mathbb{R} is the real line.

The function $f(t, c_t, V_t, \omega)$ corresponds to a felicity index at time t , and A corresponds to a measure of absolute risk aversion (of the Arrow-Pratt type) for the agent. In addition to current consumption c_t , the function f depends on utility V_t , and it may also depend on time t as well as the state of the world $\omega \in \Omega$.

The utility process V for a given consumption process c , satisfying $V_T = 0$, is given by the representation

$$V_t = E_t \left\{ \int_t^T (f(t, c_s, V_s) - \frac{1}{2} A(V_s) Z(s)' Z(s)) ds \right\}, \quad t \in [0, T] \quad (1)$$

where $E_t(\cdot)$ denotes conditional expectation given \mathcal{F}_t and $Z(t)$ is an \mathbb{R}^d -valued square-integrable progressively measurable volatility process, to be determined in our analysis. Here d is the dimension of the Brownian motion B_t . V_t is the remaining utility for c at time t , conditional on current information \mathcal{F}_t , and $A(V_t)$ is penalizing for risk.

Recall the *time-less* situation with a mean zero risk X having variance σ^2 , where the certainty equivalent m is defined by $Eu(w + X) := u(w - m)$ for a constant wealth w . Then the Arrow-Pratt approximation to m , valid for "small" risks, is given by $m \approx \frac{1}{2} A(w) \sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u . We would expect this analogy to work well in a continuous-time model with Brownian driven uncertainty.

If, for each consumption process c_t , there is a well-defined utility process V , the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z(t)dB_t$ is assumed to be a martingale, (1) has the stochastic differential equation representation

$$dV_t = \left(-f(t, c_t, V_t) + \frac{1}{2}A(V_t) Z(t)'Z(t) \right) dt + Z(t)dB_t. \quad (2)$$

If terminal utility different from zero is of interest, like for applications to e.g., life insurance, then V_T may be different from zero. U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption. A may also depend on time t .

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency, in the sense of Johnsen and Donaldson (1985), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009a)).

In this paper we consider two specifications: The first has the Kreps-Porteus utility representation, which corresponds to the aggregator of a CES type

$$f_1(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}} \quad \text{and} \quad A_1(v) = \frac{\gamma}{v} \quad (3)$$

If, for example, $A_1(v) = 0$ for all v , this means that the recursive utility agent is risk neutral. This is the main version that we analyze.

The parameters are assumed to satisfy $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma \neq 1$ (when $\rho = 1$ or $\gamma = 1$ logarithms apply). The elasticity of intertemporal substitution in consumption is denoted by $\psi = 1/\rho$. The parameter ρ we refer to as the *time* preference parameter. The version (3) yields the desired disentangling of γ from ρ .

An ordinally equivalent specification can be derived as follows. When the aggregator (f_1, A_1) is given corresponding to the utility function U_1 , there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $U_2 = \varphi \circ U_1$ has the aggregator (f_2, A_2) where

$$f_2(c, v) = ((1-\gamma)v)^{-\frac{\gamma}{1-\gamma}} f_1(c, ((1-\gamma)v)^{\frac{1}{1-\gamma}}), \quad A_2 = 0.$$

The function φ is given by

$$U_2 = \frac{1}{1-\gamma} U_1^{1-\gamma}, \quad (4)$$

for the Kreps-Porteus specification. It has has the CES-form

$$f_2(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \quad A_2(v) = 0. \quad (5)$$

The reduction to a normalized aggregator $(f_2, 0)$ does not mean that intertemporal utility is risk neutral, or that the representation has lost the ability to separate risk aversion from substitution (see Duffie and Epstein(1992a)). The corresponding utility U_2 retains the essential features, namely that of (partly) disentangling intertemporal elasticity of substitution from risk aversion. This is the (standard) version analyzed previously by Duffie and Epstein (1992a) using dynamic programming.

The normalized version is used to prove existence and uniqueness of the solution to the BSDE (2), see Duffie and Epstein (1992b) and Duffie and Lions (1992).

It is instructive to recall that the conventional additive and separable utility has aggregator

$$f(c, v) = u(c) - \delta v, \quad A = 0. \quad (6)$$

in the present framework (an ordinally equivalent one). As can be seen, even if $A = 0$, the agent of the conventional model is not risk neutral.

2.1 Homogeneity

The following result will be made use of below. For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \left(-f(t, c_t, V_t^{(c)}) + \frac{1}{2}A(V_t^{(c)}) Z(t)^{\prime(c)} Z(t)^{(c)} \right) dt + Z(t)^{(c)} dB_t \\ V_T^{(c)} = 0 \end{cases} \quad (7)$$

Theorem 1 *Assume that, for all $\lambda > 0$,*

(i) $\lambda f(t, c, v) = f(t, \lambda c, \lambda v); \forall t, c, v, \omega$

(ii) $A(\lambda v) = \frac{1}{\lambda}A(v); \forall v$

Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)} \text{ and } Z_t^{(\lambda c)} = \lambda Z_t^{(c)}, \quad t \in [0, T]. \quad (8)$$

Proof By uniqueness of the solution of the BSDEs of the type (7), all we need to do is to verify that the triple $(\lambda V_t^{(c)}, \lambda Z_t^{(c)}, \lambda K_t(\cdot)^{(c)})$ is a solution of the BSDE (7) with c_t replaced by λc_t , i.e. that

$$\begin{cases} d(\lambda V_t^{(c)}) = \left(-f(t, \lambda c_t, \lambda V_t^{(c)}) + \frac{1}{2}A(\lambda V_t^{(c)}) \lambda Z(t)^{\prime(c)} \lambda Z(t)^{(c)} \right) dt \\ + \lambda Z(t)^{(c)} dB_t; \quad 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \quad (9)$$

By (i) and (ii) the BSDE (9) can be written

$$\begin{cases} \lambda dV_t^{(c)} = \left(-\lambda f(t, c_t, V_t^{(c)}) + \frac{1}{2} \frac{1}{\lambda} A(V_t^{(c)}) \lambda^2 Z(t)^{\prime(c)} Z(t)^{(c)} \right) dt \\ + \lambda Z(t)^{(c)} dB_t; & 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \quad (10)$$

But this is exactly the equation (7) multiplied by the constant λ . Hence (10) holds and the proof is complete. \square

Remarks 1) Note that the system need not be Markovian in general, since we allow

$$f(t, c, v, \omega); (t, \omega) \in [0, T] \times \Omega$$

to be an adapted process, for each fixed c, v .

2) Similarly, we can allow A to depend on t as well ³.

Corollary 1 Define $U(c) = V_0^{(c)}$. Then $U(\lambda c) = \lambda U(c)$ for all $\lambda > 0$.

Notice that the aggregator in (3) satisfies the assumptions of the theorem.

3 The First Order Conditions

In the following we solve the consumer's optimization problem. The consumer is characterized by a utility function U and an endowment process e . For any of the versions $i = 1, 2$ formulated in the previous section, the representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to

$$E \left\{ \int_0^T c_t \pi_t dt \right\} \leq E \left\{ \int_0^T e_t \pi_t dt \right\}.$$

Here $V_t = V_t^c$ and (V_t, Z_t) is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, c_t, V_t, Z(t)) dt + Z(t) dB_t \\ V_T = 0. \end{cases} \quad (11)$$

Notice that (11) covers both the versions (3) and (5), where

$$\tilde{f}(t, c_t, V_t, Z(t)) = f_i(c_t, V_t) - \frac{1}{2} A_i(V_t) Z(t)' Z(t), \quad i = 1, 2.$$

³although not standard in Economics.

Existence and uniqueness of solutions of the BSDE is treated in the general literature on this subject. For a reference see Theorem 2.5 in Øksendal and Sulem (2013), or Hu and Peng (1995). For the equation (11) existence and uniqueness follows from Duffie and Lions (1992).

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(c; \alpha) = U(c) - \alpha E \left(\int_0^T \pi_t (c_t - e_t) dt \right).$$

Important is here that the quantity $Z(t)$ is part of the solution of the BSDE. Later we show how market clearing will finally determine the corresponding quantity in the market portfolio as a function of Z and the volatility σ_c of the growth rate of aggregate consumption. This internalizes prices in equilibrium.

In order to find the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional (Frechet) derivatives in function space, or the stochastic maximum principle. Neither of these principles require any Markovian structure of the economy. The problem is well posed since U is increasing and concave and the constraint is convex. In maximizing the Lagrangian of the problem, we can calculate the directional derivative $\nabla U(c; h)$, alternatively denoted by $(\nabla U(c))(h)$, where $\nabla U(c)$ is the gradient of U at c . Since U is continuously differentiable, this gradient is a linear and continuous functional, and thus, by the Riesz representation theorem, it is given by an inner product. This we return to in Section 5.3.

Because of the generality of the problem, let us here utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), Hu and Peng (1995), or Peng (1990)): We then have a forward/backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (11). The Hamiltonian for this problem is

$$H(t, c, v, z, y) = y_t \tilde{f}(t, c_t, v_t, z_t) - \alpha \pi_t (c_t - e_t), \quad (12)$$

where y_t is the adjoint variable. Sufficient conditions for an optimal solution to the stochastic maximum principle can be found in the literature, see e.g., Theorem 3.1 in Øksendal and Sulem (2013). Hu and Peng (1995) also study existence and uniqueness of the solution to coupled FBSDE. A unique solution exist in the present case provided there is a unique solution to the BSDE (11); again Duffie and Lions (1992) is the appropriate reference.

The adjoint equation is

$$\begin{cases} dY_t = Y(t) \left(\frac{\partial \tilde{f}}{\partial v}(t, c_t, V_t, Z(t)) dt + \frac{\partial \tilde{f}}{\partial z}(t, c_t, V_t, Z(t)) dB_t \right) \\ Y_0 = 1. \end{cases} \quad (13)$$

If c^* is optimal we therefore have

$$Y_t = \exp\left(\int_0^t \left\{ \frac{\partial \tilde{f}}{\partial v}(s, c_s^*, V_s, Z(s)) - \frac{1}{2} \left(\frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s)) \right)^2 \right\} ds + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s)) dB(s)\right) \quad a.s. \quad (14)$$

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y \frac{\partial \tilde{f}}{\partial c}(t, c^*, v, z) - \alpha \pi = 0$$

or

$$\alpha \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial c}(t, c_t^*, V(t), Z(t)) \quad a.s. \quad \text{for all } t \in [0, T]. \quad (15)$$

Notice that the state price deflator π_t at time t depends, through the adjoint variable Y_t , on the entire optimal paths (c_s, V_s, Z_s) for $0 \leq s \leq t$. (The economy may be allowed to be non-Markovian since $\tilde{f}(\cdot)$ may also be allowed to depend on the state of nature.)

When $\gamma = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator (6) of the conventional model, so the state price deflator is a Markov process, the utility is additive and dynamic programming is often used. (To actually solve the associated Bellman equation in continuous time models, most of the coefficients (volatilities) must be assumed to be constants.)

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility V_t at time t is optimal.

We now have the first order conditions for both the versions of recursive utility outlined in Section 3. We analyze the non-ordinal version, denoted Model 1, with aggregator given by (3). The ordinally equivalent version (5) is analyzed in the Appendix.

4 The financial market

Having established the general recursive utility of interest, in this section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model.

Let $\nu(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t , and let $\sigma(t)$ denote the matrix of diffusion coefficients of the risky asset prices, normalized

by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for asset returns. Both $\nu(t)$ and $\sigma(t)$ are progressively measurable, ergodic processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \quad (16)$$

subject to the intertemporal budget constraint

$$dW_t = (W_t(\varphi'_t \cdot \nu(t) + r_t) - c_t)dt + W_t\varphi'_t \cdot \sigma(t)dB_t. \quad (17)$$

Here $\varphi'_t = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities.

Market clearing requires that $\varphi'_t\sigma(t) = (\delta_t^M)'\sigma(t) = \sigma_M(t)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market portfolio, and δ_t^M are the fractions of the different securities, $j = 1, \dots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

The model is a pure exchange economy where the aggregate consumption process c_t in society is exogenously given, and the single agent optimally consumes $c_t = e_t$ in every period, i.e., the agent optimally consumes the endowment process e_t at every date t . The main issue is then the determination of prices, including risk premiums and the interest rate, consistent with this behavior.

In the above we have interpreted the market portfolio as a proxy for the wealth portfolio, a common assumption in settings like this. This may, however be inaccurate. We return to this in Section 7.

5 The analysis for the nonordinal model

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. The first order conditions are

$$\alpha \pi_t = Y_t \frac{\partial f_1}{\partial c}(c_t, V_t) \quad \text{a.s. for all } t \in [0, T] \quad (18)$$

where f_1 is given in (3). The volatility $Z(t)$ and the utility process V_t satisfy the following dynamics

$$dV_t = \left(-\frac{\delta}{1-\rho} \frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho}} + \frac{1}{2} \frac{\gamma}{V_t} Z'(t)Z(t) \right) dt + Z(t)dB_t \quad (19)$$

where $V(T) = 0$. This is the backward equation.

Aggregate consumption is exogenous, with dynamics of the form

$$\frac{dc_t}{c_t} = \mu_c(t) dt + \sigma_c(t) dB_t, \quad (20)$$

where $\mu_c(t)$ and $\sigma_c(t)$ are measurable, \mathcal{F}_t adapted stochastic processes, satisfying appropriate integrability properties. We assume these processes to be ergodic, so that we may 'replace' (estimate) time averages by state averages.

The function \tilde{f} of Section 4 is given by

$$\tilde{f}(t, c, v, z) = f_1(c, v) - \frac{1}{2}A(v)z'z,$$

and since $A(v) = \gamma/v$, from (13) the adjoint variable Y has dynamics

$$dY_t = Y_t \left(\left\{ \frac{\partial}{\partial v} f_1(c_t, V_t) + \frac{1}{2} \frac{\gamma}{V_t^2} Z'(t)Z(t) \right\} dt - A(V_t)Z(t) dB_t \right), \quad (21)$$

where $Y(0) = 1$. From the FOC in (45) we obtain the dynamics of the state price deflator. We now use the notation f for f_1 for simplicity (except in the Appendix, where f is f_2). We also use the notation $Z(t)/V(t) = \sigma_V(t)$, valid for $V \neq 0$. By Theorem 1 the term $\sigma_V(t)$ is homogeneous of order zero in c .

We then seek the connection between V_t and $\sigma_V(t)$ and the rest of the economy. Notice that Y is not a bounded variation process, and by Ito's lemma

$$d\pi_t = f_c(c_t, V_t) dY_t + Y_t df_c(c_t, V_t) + dY_t df_c(c_t, V_t). \quad (22)$$

By the adjoint and the backward equations this is

$$\begin{aligned} d\pi_t &= Y_t f_c(c_t, V_t) \left(\left\{ f_v(c_t, V_t) + \frac{1}{2} \gamma \sigma_V'(t) \sigma_V(t) \right\} dt - \gamma \sigma_V(t) dB_t \right) \\ &+ Y_t \frac{\partial f_c}{\partial c}(c_t, V_t) dc_t + Y_t \frac{\partial f_c}{\partial v}(c_t, V_t) dV_t + dY_t df_c(c_t, V_t) \\ &+ Y_t \left(\frac{1}{2} \frac{\partial^2 f_c}{\partial c^2}(c_t, V_t) (dc_t)^2 + \frac{\partial^2 f_c}{\partial c \partial v}(c_t, V_t) (dc_t)(dV_t) + \frac{1}{2} \frac{\partial^2 f_c}{\partial v^2}(c_t, V_t) (dV_t)^2 \right). \end{aligned} \quad (23)$$

Here

$$\begin{aligned} f_c(c, v) &:= \frac{\partial f(c, v)}{\partial c} = \delta c^{-\rho} v^\rho, \quad f_v(c, v) := \frac{\partial f(c, v)}{\partial v} = -\frac{\delta}{1-\rho} (1 - \rho c^{1-\rho} v^{\rho-1}), \\ \frac{\partial f_c(c, v)}{\partial c} &= -\delta \rho c^{-(1+\rho)} v^\rho, \quad \frac{\partial f_c(c, v)}{\partial v} = \delta \rho v^{\rho-1} c^{-\rho}, \end{aligned}$$

$$\frac{\partial^2 f_c}{\partial c^2}(c, v) = \delta\rho(\rho + 1)v^\rho c^{-(\rho+2)}, \quad \frac{\partial^2 f_c}{\partial c \partial v}(c, v) = -\delta\rho^2 v^{\rho-1} c^{-(\rho+1)},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c, v) = \delta\rho(\rho - 1)v^{\rho-2} c^{-\rho}.$$

5.1 The risk premiums

Denoting the dynamics of the state price deflator by

$$d\pi_t = \mu_\pi(t) dt + \sigma_\pi(t) dB_t, \quad (24)$$

from (23) and the above expressions we obtain the drift and the diffusion terms of π_t as

$$\begin{aligned} \mu_\pi(t) = \pi_t & \left(-\delta - \rho\mu_c(t) + \frac{1}{2}\rho(\rho + 1)\sigma'_c(t)\sigma_c(t) \right. \\ & \left. + \rho(\gamma - \rho)\sigma'_c(t)\sigma_V(t) + \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma'_V(t)\sigma_V(t) \right) \end{aligned} \quad (25)$$

and

$$\sigma_\pi(t) = -\pi_t(\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t)) \quad (26)$$

respectively.

Notice that π_t is not a Markov process since $\mu_\pi(t)$ and $\sigma_\pi(t)$ depend on π_t , and the latter variable depends on consumption and utility from time zero to time t .

Interpreting π_t as the price of the consumption good at time t , by the first order condition it is a decreasing function of consumption c since $f_{cc} < 0$.

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t} \sigma_\pi(t) \sigma_R(t). \quad (27)$$

It follows immediately from (26) and (27) that the formula for the risk premium of any risky security R is

$$\mu_R(t) - r_t = \rho\sigma_c(t)\sigma_R(t) + (\gamma - \rho)\sigma_V(t)\sigma_R(t). \quad (28)$$

This is our basic result for risk premiums.

It remains to connect $\sigma_V(t)$ to observables in the economy, which we do below. Before that we turn to the interest rate.

5.2 The equilibrium interest rate

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (29)$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use (25). We then obtain the following

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho + 1)\sigma'_c(t)\sigma_c(t) - \rho(\gamma - \rho)\sigma_{cV}(t) - \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma'_V(t)\sigma_V(t). \quad (30)$$

This is our basic result for the equilibrium short rate.

The potential for these two relationships to solve the puzzles should be apparent. We return to a discussion later.

We proceed to link the volatility term $\sigma_V(t)$ to observable quantities in the market that can be estimated from market data.

5.3 The determination of the volatility of the market portfolio.

In order to determine $\sigma_M(t)$ from the primitives $\sigma_V(t)$ and $\sigma_c(t)$, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t \left(\int_t^T \pi_s c_s ds \right), \quad (31)$$

where c is optimal. From Theorem 1 it follows that the non-ordinal utility function U is homogenous of degree one. By the definition of directional derivatives we have that

$$\begin{aligned} \nabla U(c; c) &= \lim_{\alpha \downarrow 0} \frac{U(c + \alpha c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{U(c(1 + \alpha)) - U(c)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{(1 + \alpha)U(c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\alpha U(c)}{\alpha} = U(c), \end{aligned}$$

where the third equality uses that U is homogeneous of degree one. By the Riesz representation theorem it follows from the linearity and continuity of

the directional derivative that, by the first order condition

$$\nabla U(c; c) = E \left(\int_0^T \pi_t c_t^* dt \right) = W_0 \pi_0 \quad (32)$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (31) for $t = 0$. Thus $U(c) = \pi_0 W_0$.

Let $V_t = V_t^{(c)}$ denote future utility at the optimal consumption for our representation. Since also V_t is homogeneous of degree one and continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic relationship holds here for the associated directional derivatives at any time t , i.e.,

$$\nabla V_t(c; c) = E_t \left(\int_t^T \pi_s^{(t)} c_s ds \right) = V_t(c)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t . As for the discrete time model, it follows by results in Skiadas (2009a) that with assumption A2, implying that this quantity is independent of past consumption, the consumption history in the adjoint variable Y_t is 'removed' from the state price deflator π_t , so that $\pi_s^{(t)} = \pi_s / Y_t$ for all $t \leq s \leq T$. By this it follows that

$$V_t = \frac{1}{Y_t} \pi_t W_t. \quad (33)$$

This connects the dynamics of V to the rest of the economy. By the product rule,

$$dV_t = d(Y_t^{-1})(\pi_t W_t) + Y_t^{-1} d(\pi_t W_t) + dY_t^{-1} d(\pi_t W_t). \quad (34)$$

where

$$d(\pi_t W_t) = W_t d\pi_t + \pi_t dW_t + d\pi_t dW_t \quad (35)$$

Ito's lemma gives

$$d\left(\frac{1}{Y_t}\right) = \left(-\left(\frac{1}{Y_t}\right)(f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t) + \frac{\gamma^2}{Y_t}\sigma_V'(t)\sigma_V(t))dt + \frac{1}{Y_t}\gamma\sigma_V(t)dB_t\right) \quad (36)$$

From the equations (34)-(36) it follows by the market clearing condition $\varphi_t' \cdot \sigma(t) = \sigma_M(t)$ that

$$V_t \sigma_V(t) = \frac{1}{Y_t} \left(\pi_t W_t \gamma \sigma_V + \pi_t W_t \sigma_M(t) - \pi_t W_t (\rho \sigma_c(t) + (\gamma - \rho) \sigma_V(t)) \right) \quad (37)$$

From the expression (33) for V_t we obtain the following equation for σ_V

$$\sigma_V(t) = \gamma\sigma_V(t) + \sigma_M(t) - (\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t))$$

from which it follows that

$$\sigma_M(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t). \quad (38)$$

This is the internalization of W_t : The volatility of the market portfolio is a linear sum of the volatility of future utility and the volatility of the growth rate of aggregate consumption, both parts of the primitives of the economic model.

This relationship can now be used to express $\sigma_V(t)$ in terms of the other two volatilities as

$$\sigma_V(t) = \frac{1}{1 - \rho}(\sigma_M(t) - \rho\sigma_c(t)). \quad (39)$$

Alternatively, and somewhat easier, we can use the relation $V_t Y_t = \pi_t W_t$ and the product rule directly to find these results.

Inserting the expression (39) into (28) and (30) we obtain the risk premiums

$$\mu_R(t) - r_t = \frac{\rho(1 - \gamma)}{1 - \rho} \sigma'_c(t)\sigma_R(t) + \frac{\gamma - \rho}{1 - \rho} \sigma'_M(t)\sigma_R(t), \quad (40)$$

and the short rate

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1 - \gamma\rho)}{1 - \rho} \sigma_c(t)'\sigma_c(t) + \frac{1}{2} \frac{\rho - \gamma}{1 - \rho} \sigma'_M(t)\sigma_M(t) \quad (41)$$

respectively.

The expression for the risk premium was derived by Duffie and Epstein (1992a) based on dynamic programming, assuming the volatilities involved to be constants. The expression for the real interest rate is new to this paper.

The version treated by Duffie and Epstein (1992a) is the ordinally equivalent one based on (5), which was claimed to be better suited for dynamic programming. We show in the Appendix that under the assumptions of this paper the results are the same for both versions of recursive utility.

6 Summary of the model

Taking existence of equilibrium as given, the main results in this section are summarized as

Theorem 2 *For the non-ordinal model specified in Sections 2-5, in equilibrium the risk premium of any risky asset is given by (40) and the real interest rate by (41).*

The resulting risk premiums are linear combinations of the consumption-based CAPM and the market-based CAPM at each time t . The original derivation of the CAPM as an equilibrium model was given by Mossin (1966). His derivation was in a *time-less* setting, where the interest rate plays no role.

When the time preference $\rho = 0$ in Theorem 2, only the market-based CAPM remains. Accordingly, this model can be considered a *dynamic* version of the market-based CAPM, with the associated interest rate given by (41). In the present setting with recursive utility we denote this model by CAPM++. Below we also calibrate this version to the data summarized in Table 1 below.

The last two terms in the short rate has, together with the expression for the equity premium, the potential to explain the low, observed values of the real rate. These terms are also interesting when it comes to *precautionary savings*. It is easy to see that when $\rho < 1$, then precautionary savings result when $\frac{1}{\rho} > \gamma > \rho$: An increase in the variability of the consumption growth rate, or the variability of the wealth portfolio, both lead to a decrease in the real interest rate r_t . This would be the analogue of a *prudent agent* in the conventional model. This also seems a plausible interrelationship between these two parameters for another reason: When $\gamma > \rho$ the agent prefers early resolution of uncertainty to late (see Fig. 1).

The risk premium decreases as $\sigma_c(t)$ increases when $\gamma > \rho$ and $\rho < 1$. The conventional model can only predict an increase in the risk premium when this volatility increases. When $\sigma_M(t)$ increases in this situation, the interest rate decreases and the risk premium increases. The same happens if $\gamma < \rho$ and $\rho > 1$. The conventional model has no answers for this.

6.1 Calibrations

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by M , as well as for the annualized consumption data, denoted c , and the government bills, denoted b ⁴.

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient⁵.

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This

⁴There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

⁵The full data set was provided by Rajnish Mehra.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M, c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M, b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c, b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

gives, e.g., the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa(t)$. The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

First we interpret the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index. The conventional, additive Eu-model we obtain from (40) and (41) when $\gamma = \rho$. We then have two equations in two unknowns which provide estimates for the preference parameters by the "method of moments"⁶. The result for the Eu-model is $\gamma = 26.3$ and $\delta = -.015$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%. This is the equity premium puzzle.

If we insist on a nonnegative impatience rate, as we probably should (but see Kocherlakota (1990)), this means that the real interest rate explained by the model is larger than 3.3% (when $\delta = .01$, say) for the period considered, but it is estimated, as is seen from Table 2, to be less than one per cent. The EIS parameter is calibrated to $\psi = .037$, which is considered to be too low for the representative individual.

There is of course some sampling error, so that these estimates are not exact, but clearly indicates that something is wrong with this model.

⁶Indeed, what we really do here is to use the assumption about ergodicity of the various μ_t and σ_t processes. This enables us to replace "state averages" by "time averages", the latter being given in Table 2.

Calibrations of the model (40) and (41) are presented in Table 3, for plausible ranges of the parameters. We have consider Government bills as risk free.

	γ	ρ	EIS	δ
Standard Model	26.37	26.37	.037	-.014
$\delta = .010$ fixed	.005	1.61	.62	.010
$\delta = .015$ fixed	.460	1.34	.74	.015
$\delta = .020$ fixed	.900	1.06	.94	.020
$\delta = .023$ fixed	1.15	.90	1.11	.023
$\delta = .030$ fixed	1.74	.48	2.08	.030
$\delta = .035$ fixed	2.14	.18	5.56	.035
$\rho = .90$ fixed	1.15	.90	1.11	.023
$\rho = .80$ fixed	1.30	.80	1.25	.025
$\rho = .50$ fixed	1.72	.50	2.00	.030
$\rho = .40$ fixed	1.86	.40	2.50	.031
CAPM ++	2.38	0.00	$+\infty$.038
$\gamma = 0.50$ fixed	0.50	1.31	0.79	.015
$\gamma = 1.05$ fixed	1.05	.97	1.03	.040
$\gamma = 1.50$ fixed	1.50	.66	1.51	.027
$\gamma = 2.00$ fixed	2.00	.30	3.33	.033
$\gamma = 2.30$ fixed	2.30	.07	14.30	.036

Table 3: Various Calibrations Consistent with Table 2

As noticed, ρ can be constrained to be zero, in which case the model reduces to what we have called the CAPM++:

$$\mu_R(t) - r_t = \gamma \sigma_{M,R}(t), \quad r_t = \delta - \frac{\gamma}{2} \sigma'_M(t) \sigma_M(t).$$

The risk premium is that of the ordinary CAPM-type, while the interest rate is new. This version of the model corresponds to "neutrality" of consumption transfers. Also, from the expression for the interest rate we notice that the short rate decreases in the presence of increasing market uncertainty. Solving these two non-linear equations consistent with the data of Table 2, we obtain

$$\gamma = 2.38 \quad \text{and} \quad \delta = .038.$$

In the conventional model this simply gives risk neutrality, i.e., $\gamma = \rho = 0$, so this model gives a risk premium of zero, and a short rate of $r = \delta$.

The original equilibrium model developed by Mossin (1966) was in a one period (a time-less) setting with consumption only on the terminal time

point, in which case wealth equals consumption. Since there was no consumption on the initial time, no intertemporal aspects of consumption transfers arose in the classical model. This naturally corresponds to $u(c) = c$ for the the felicity index regarding consumption transfers, meaning $\rho = 0$ and $\psi = 1/\rho = +\infty$, and corresponding to perfect substitutability of consumption across time.

When the instantaneous correlation coefficient $\kappa_{Mc}(t)$ of returns and the aggregate consumption growth rate is small, our model handles this situation much better than the conventional one. The extreme case when $\kappa_{Mc}(t) = 0$ is, for example, consistent with the solution presented above for $\rho = 0$, which gives reasonable parameter values for the other parameters.

Most of the plausible calibration points presented in Table 3 correspond to $\gamma > 1 > \rho$ and accordingly $EIS > 1$, for the data summarized in Table 2. Accordingly, these are located in the early resolution part of the (ρ, γ) -plane where $\gamma > \rho$.

However, the present version is also consistent with calibrations in the region $0 < \gamma < \rho < 1$, corresponding to late resolution. As an example, if $\rho = 1.1$, this is consistent with $\delta = .02$ and $\gamma = .90$. The square root utility function is used in many examples in various textbooks (for the conventional model). For $\gamma = .5$ the model calibrates to $\delta = .015$ and $\rho = 1.31$, i.e., late resolution but otherwise for reasonable values of the parameters (calibration point Calibr 2 in Fig. 1).

6.2 Some new features of the model

It is reassuring that the risk premium of any risky asset depends on other investment opportunities in the financial market, and not just on this asset's covariance rate with consumption.

It is also satisfying that the return rate on government bonds depend on more than just the growth rate and the variance rate of aggregate consumption, but also on characteristics of other investment opportunities in the financial market.

Faced with increasing consumption uncertainty, the 'prudent' consumer will save and the interest rate accordingly falls in equilibrium (this is a fruit-tree economy). This is precautionary savings, and takes place if $(1 - \rho\gamma)/(1 - \rho) > 0$, which then becomes the natural definition of prudence for this version of recursive utility. As typical examples, the calibration point Calibr 1 in Fig. 1 satisfies this requirement, as does the point CAPM++.

When the uncertainty of the return of the market portfolio increases, the interest rate decreases provided $\gamma > \rho$ and $\rho < 1$, or if $\gamma < \rho$ and $\rho > 1$, and otherwise increases. The point Calibr 1 satisfies the first of

these requirements, so does the point CAPM++, while Calibr 2 satisfies the second.

If $\sigma_M(t)$ increases, the equity premium increases and the interest rate decreases when $\gamma > \rho$ and $\rho < 1$.

This kind of discussion has no place in the conventional model, since when $\rho = \gamma$ there is no direct connection to the securities market (nor to the wealth portfolio) in the expression for the risk premium (40). Similarly, the interest rate in (41) has no connection to the wealth portfolio in the conventional model.

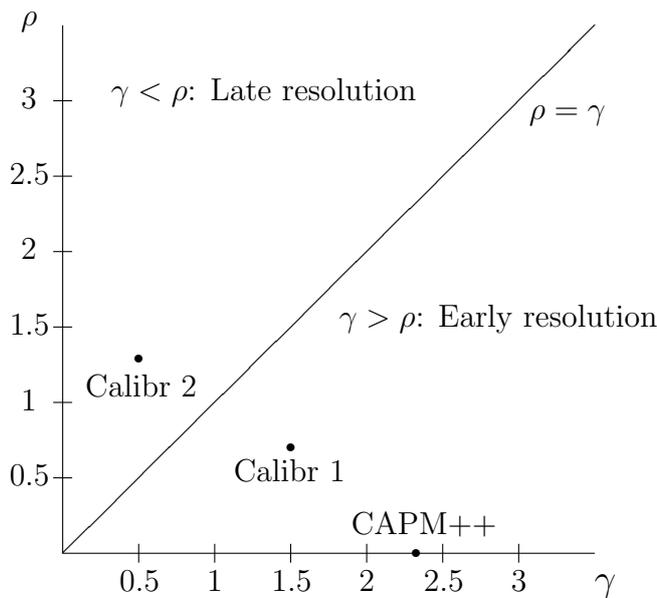


Figure 1: Calibration points in the (γ, ρ) -space

The discrete-time recursive model of Epstein and Zin (1989-91) is the one that is mostly used in applications. This model does not calibrate as well as the model of the present paper. In the discrete-time model certain approximations are usually made to get risk premiums and the interest rate in a simple form. As a result of these approximations, in particular the ones made for the interest rate, the calibrations only become reasonable when the impatience rate approaches 10 per cent. We conjecture that without these

approximations the results would be more in line with the ones in the present paper.

6.3 Government bills

In the above discussion we have interpreted Government bills as risk free. With this in mind, there is another problem with the conventional, additive Eu-model. From Table 2 we see that there is a negative correlation between Government bills and the consumption growth rate. Similarly there is a positive correlation between the return on S&P-500 and Government bills.

If we interpret Government bills as risk free, the former correlation should be zero for the CCAPM-model to be consistent. Since this correlation is not zero, then γ must be zero, which is inconsistent with the the above (and the model).

The Government bills used by Mehra and Prescott (1985) have duration one month, and the data are yearly, in which case Government bills are not the same as Sovereign bonds with duration of one year. One month bills in a yearly context will then contain price risk 11 months each year, and hence the estimate of the real, risk free rate is, perhaps, strictly lower than 0.80%. Whatever the positive value of the risk premium is, the resulting value of γ is negative. With bills included, the conventional, Eu-model does not seem to have enough 'degrees of freedom' to match the data, since in this situation the model contains three basic relationships and only two 'free parameters'.

The recursive model does much better in this regard, and yields more plausible results as it has enough degrees of freedom for this problem.

Exactly what risk premium bills command we can here only stipulate. For a risk premium of .0040 for the bills we have a third equation, namely

$$\mu_b(t) - r_t = \frac{\rho(1 - \gamma)}{1 - \rho} \sigma_{c,b}(t) + \frac{\gamma - \rho}{1 - \rho} \sigma_{M,b} \quad (42)$$

to solve together with the equations (40) and (41). With the covariance estimates provided in Table 2, we have three equations in three unknowns, giving the following values $\delta = 0.027$, $\gamma = 1.76$ and $\rho = 0.53$. This risk premium of the bills indicates that the estimate of the real rate is only .0040, which may be a bit low, but these results are far better than the conventional, additive Eu-model can provide.

This may have several important consequences. To mention just one, recall the controversy around the Stern report, in which an estimate of 1.4 per cent for the real rate is suggested. Stern (2007) set the impatience rate $\rho = 0.001$, and received critique for this as well. Based on the above, the real

rate could have been set close to zero for climate related projects, and still have good model, and empirical support.

7 The market portfolio is not a proxy for the wealth portfolio

In the paper we have focused on comparing two models, assuming the market portfolio can be used as a proxy for the wealth portfolio. Suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a pretty good impression of how the two models compare, which we now attempt.

In the conventional model with constant coefficients the growth rate of the wealth portfolio has the same volatility as the growth rate of aggregate consumption. Taking this quantity as the lower bound for this volatility, we indicate how the models compare when the market portfolio can not be taken as a proxy for the wealth portfolio. Below we first set $\sigma_W(t) = .05$, $\kappa_{c,W} = .40$ as before, and set $\kappa_{W,R} = .70$. The model can be written

$$\mu_M(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho} \sigma'_c(t)\sigma_M(t) + \frac{\gamma-\rho}{1-\rho} \sigma'_W(t)\sigma_M(t), \quad (43)$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma'_c(t)\sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma'_W(t)\sigma_W(t). \quad (44)$$

Here M stands for the market portfolio and W for the wealth portfolio, so that (43) is the equity premium. The calibrations are given in Table 4. The results are in favor of low values of the impatience rate δ . Typical values of γ fall between 2.7 and 4.5. The CAPM++ results when $\rho = 0$, and is here consistent with a fairly high value of $\gamma = 10.8$, but with a reasonable impatience rate of 1.7 per cent.

This value of .05 for the volatility of the wealth portfolio may be somewhat low. A more reasonable one is likely to be somewhere in between $\sigma_c(t)$ and $\sigma_M(t)$, so we suggest $\sigma_W(t) = .10$. We stipulate the correlation coefficient $\kappa_{W,R} = .80$, and maintain the estimate of $\kappa_{c,W} = .40$. Calibrations under these assumptions are given in Table 5. As can be seen from the table, there is now a wide range of plausible solutions.

Parameters	γ	ρ	EIS	δ
Recursive model				
$\delta = 10^{-6}$	2.73	.91	1.10	10^{-6}
$\delta = .001$	3.30	.88	1.36	.001
$\delta = .010$	7.86	.49	2.04	.010
$\delta = .015$	9.92	.18	5.55	.015
$\rho = 0.00$ CAPM++	10.80	.00	$+\infty$.017
$\rho = .90$	3.01	.90	1.11	.000
$\rho = .85$	3.85	.85	1.18	.002
$\rho = .80$	4.60	.80	1.25	.003
$\rho = .70$	5.89	.70	1.43	.006
$\gamma = 2.80$	2.80	.91	1.10	.000
$\gamma = 3.50$	3.50	.87	1.14	.001
$\gamma = 4.00$	4.00	.84	1.19	.002
$\gamma = 4.50$	4.50	.81	1.23	.003

Table 4: Calibrations of the model when $\sigma_W(t) = .05$, $\kappa_{W,R} = .70$ and $\kappa_{e,W} = .40$.

The illustrations in this section give a fairly clear indication of how the model performs when the market portfolio is not a proxy for the wealth portfolio. Many additional examples could of course be given, and the model can be extended and moved in various directions, as indicated by the extant literature. However, the examples presented are fairly simple, and give a reasonable illustration of how the recursive model behaves. Compared to the conventional model the difference is dramatic. In solving puzzles, as few features as possible should be altered at the time in order to discover what made the difference.

8 Extensions

The recursive models analyzed in this paper has been extended to include jump dynamics (Aase (2015)), which may be of particular interest in modeling stock market movements. This approach allows for an additional parameter γ_0 for risk aversion related to jump size risk, which can be different from γ . A discrete time version has also been considered, and the results are comparable (Aase (2013)). Also a heterogeneous model in continuous-time give comparable results (Aase (2014)).

Parameters	γ	ρ	EIS	δ
Recursive model				
$\delta = .015$	0.93	1.01	.99	.015
$\delta = .020$	2.11	0.74	1.35	.020
$\delta = .025$	3.26	0.44	2.27	.025
$\delta = .030$	4.37	0.11	9.09	.030
$\rho = .000$	4.72	0.00	$+\infty$.032
$\rho = .200$	4.08	0.20	5.00	.029
$\rho = .300$	3.75	0.30	3.33	.027
$\rho = .500$	3.04	0.50	2.00	.024
$\rho = .800$	1.87	0.80	1.25	.019
$\rho = .900$	1.44	0.90	1.11	.017
$\rho = .950$	1.22	0.95	1.05	.016
$\gamma = 1.01$	1.01	.99	1.01	.015
$\gamma = 2.00$	2.00	.77	1.30	.020
$\gamma = 2.50$	2.50	.64	1.56	.022
$\gamma = 3.00$	3.00	.51	1.96	.024
$\gamma = 3.50$	3.50	.37	2.70	.026
$\gamma = 4.00$	4.00	.23	4.35	.028
$\gamma = 4.50$	4.50	.07	14.29	.030

Table 5: Calibrations of the model when $\sigma_W(t) = .10$, $\kappa_{W,R} = .80$ and $\kappa_{c,W} = .40$.

9 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial and macro-economics. The continuous-time recursive model is shown to fit data much better than the conventional Eu-model. Our formal approach is to use the stochastic maximum principle and forward/backward stochastic differential equations. This method can handle state dependence.

In equilibrium the stochastic process of the market portfolio (or the wealth portfolio) is determined from the stochastic processes of future utility and the the growth rate of aggregate consumption.

With this in place, the model calibrates to plausible values of the parameters under reasonable assumptions.

Recursive utility in continuous time has two ordinally equivalent versions, where one version admits an unambiguous interpretation in the Kreps-Porteus specification. This is the model we analyze in this paper. We show

that both versions have the same risk premiums and short rates.

When the market portfolio is not a proxy for the wealth portfolio, our results are still convincing, in fact, these seem the most interesting results from the calibrations.

10 Appendix

10.1 The ordinally equivalent model analyzed by the stochastic maximum principle

In this section we demonstrate how our method works for the ordinally equivalent version (5). For this model the first order conditions are given by

$$\alpha \pi_t = Y_t \frac{\partial f}{\partial c}(c_t, V_t) \quad \text{a.s. for all } t \in [0, T] \quad (45)$$

where $\tilde{f}(t, c, v, \tilde{\sigma}_v) = f_2(c, v) := f(c, v)$ is given in (5), and where the adjoint variable $Y(t)$ is

$$Y_t = \exp\left(\int_0^t \frac{\partial f}{\partial v}(c_s, V_s) ds\right) \quad \text{a.s.} \quad (46)$$

As can be noted, for this version the adjoint process is of bounded variation⁷.

The model for the aggregate consumption is the same as before, and the process V_t is assumed to follow the dynamics

$$\frac{dV_t}{(1-\gamma)V_t} = \mu_V(t) dt + \sigma_V(t) dB_t \quad (47)$$

where

$$\tilde{\sigma}_V(t) = (1-\gamma)V_t\sigma_V(t), \quad \text{and} \quad \mu_V(t) = -\frac{\delta}{1-\rho} \left(\frac{c_t^{1-\rho} - ((1-\gamma)V_t)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)V_t)^{\frac{1-\rho}{1-\gamma}}} \right).$$

Here we have called $Z_2(t) = \tilde{\sigma}_V(t)$, and $f_2 = f$ for simplicity of notation. When $\gamma > 1$ utility V is negative so the product $(1-\gamma)V > 0$ a.s., which gives us a positive volatility of V provided $\sigma_V(t) > 0$ a.e. From the FOC (45) we then get the dynamics of the state price deflator:

$$d\pi_t = f_c(c_t, V_t) dY_t + Y_t df_c(c_t, V_t). \quad (48)$$

⁷Originally the author derived this FOC using utility gradients based on a result of Duffie and Skiadas (1994).

Using Ito's lemma this becomes

$$\begin{aligned}
d\pi_t &= Y_t f_c(c_t, V_t) f_v(c_t, V_t) dt + Y_t \frac{\partial f_c}{\partial c}(c_t, V_t) dc_t + Y_t \frac{\partial f_c}{\partial v}(c_t, V_t) dV_t \\
&+ Y_t \left(\frac{1}{2} \frac{\partial^2 f_c}{\partial c^2}(c_t, V_t) (dc_t)^2 + \frac{\partial^2 f_c}{\partial c \partial v}(c_t, V_t) (dc_t)(dV_t) + \frac{1}{2} \frac{\partial^2 f_c}{\partial v^2}(c_t, V_t) (dV_t)^2 \right).
\end{aligned} \tag{49}$$

Here

$$\begin{aligned}
f_c(c, v) &:= \frac{\partial f(c, v)}{\partial c} = \frac{\delta c^{-\rho}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \\
f_v(c, v) &:= \frac{\partial f(c, v)}{\partial v} = \frac{\delta}{1-\rho} \left(c^{1-\rho} ((1-\gamma)v)^{-\frac{1-\rho}{1-\gamma}} (\rho-\gamma) + (\gamma-1) \right), \\
\frac{\partial f_c(c, v)}{\partial c} &= -\frac{\delta \rho c^{-\rho-1}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}, \quad \frac{\partial f_c(c, v)}{\partial v} = \delta(\rho-\gamma) c^{-\rho} ((1-\gamma)v)^{-\frac{1-\rho}{1-\gamma}}, \\
\frac{\partial^2 f_c}{\partial c^2}(c, v) &= \frac{\delta \rho(1+\rho) c^{-\rho-2}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \quad \frac{\partial^2 f_c}{\partial c \partial v}(c, v) = \frac{\rho \delta (\gamma-\rho) c^{-\rho-1}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}},
\end{aligned}$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c, v) = \frac{\delta (\gamma-\rho) (1-\rho) c^{-\rho}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}+1}}.$$

Denoting the dynamics of the state price deflator by

$$d\pi_t = \mu_\pi(t) dt + \sigma_\pi(t) dB_t, \tag{50}$$

from (49) and the above expressions we now have that the drift and the diffusion terms of π_t are given by

$$\begin{aligned}
\mu_\pi(t) &= Y_t \left(\frac{\delta^2}{1-\rho} (\rho-\gamma) c_t^{2(1-\rho)-1} ((1-\gamma)V_t)^{-\frac{2(1-\rho)}{1-\gamma}+1} \right. \\
&\quad \left. - \frac{(1-\gamma)\delta^2}{1-\rho} c_t^{-\rho} ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} - \delta \rho c_t^{-\rho} ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \mu_c(t) \right. \\
&\quad \left. - \delta c_t^{-\rho} (\rho-\gamma) ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}} f(c_t, V_t) + \frac{1}{2} \delta \rho (1+\rho) c_t^{-\rho} ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma'_c(t) \sigma_c(t) \right. \\
&\quad \left. - \delta \rho c_t^{-\rho} (\rho-\gamma) ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma_{cV}(t) \right. \\
&\quad \left. - \frac{1}{2} \delta (\rho-\gamma) (1-\rho) c_t^{-\rho} ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma'_V(t) \sigma_V(t) \right), \tag{51}
\end{aligned}$$

and

$$\sigma_\pi(t) = Y_t \delta c_t^{-\rho} \left((-\rho) \sigma_c(t) ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} + \right. \\ \left. (\rho-\gamma) ((1-\gamma)V_t)^{-\frac{1-\rho}{1-\gamma}} ((1-\gamma)V_t) \sigma_V(t) \right) \quad (52)$$

respectively.

10.2 The risk premium

The risk premium is as before given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t} \sigma_{R\pi}(t), \quad (53)$$

where $\sigma_{R\pi}(t)$ is the instantaneous covariance of the increments of R and π .

Combining the FOC with the result in (52), the formula for the risk premium in terms of the primitives of the model is accordingly given by

$$\mu_R(t) - r_t = \rho \sigma_{Rc}(t) + (\gamma - \rho) \sigma_{RV}(t). \quad (54)$$

This is a basic result of our analysis with recursive utility, and is seen to be the same as for the nonordinal version based on (3) in terms of $\sigma_V(t)$. We return to the equilibrium determination of this term. Before we do that, we give an expression for the equilibrium interest rate r_t .

10.3 The equilibrium interest rate

The equilibrium interest rate r_t is again given by the general formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (55)$$

In order to find an expression for r_t in terms of the primitives of the model, we use the formula for $f(c_t, V_t)$ from (5) in the expression for $\mu_\pi(t)$ in (51). We then obtain the following

$$r_t = \delta + \rho \mu_c(t) - \frac{1}{2} \rho (\rho + 1) \sigma'_c(t) \sigma_c(t) + \\ \rho (\rho - \gamma) \sigma_{cV}(t) + \frac{1}{2} (\rho - \gamma) (1 - \rho) \sigma'_V(t) \sigma_V(t). \quad (56)$$

Again, this is the second basic result of our analysis with recursive utility, and is seen to be the same as the one for the nonordinal version based on (3) in terms of $\sigma_V(t)$.

In order to link the volatility term $\sigma_V(t)$ to an observable (or estimable) quantity in the market, we use market clearing in the financial market, and properties of recursive utility.

10.4 The volatility of the market portfolio

First recall that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t \left(\int_t^T \pi_s c_s ds \right),$$

where c is optimal. Next recall the connection between the two ordinally equivalent recursive utility representations that we deal with. It follows from (4) that

$$\begin{aligned} \nabla U_2(c; c) &= U_1(c)^{-\gamma} \nabla U_1(c; c) \\ &= U_1(c)^{-\gamma} U_1(c) = U_1(c)^{1-\gamma} = (1-\gamma)U_2(c). \end{aligned} \quad (57)$$

The second equality follows as in Section 5.3 since U_1 is homogeneous of degree one. It follows from the first order condition that

$$\nabla U_2(c; c) = E \left(\int_0^T \pi_t c_t dt \right) = W_0 \pi_0.$$

Let $\tilde{V}_t(c_t)$ and $V_t(c_t)$ denote future utility at the optimal consumption for our current representation and the non-ordinal version of recursive utility, respectively.

Moving to time t , the same, basic relationship holds here for the associated directional derivatives, using the dominated convergence and the Riesz representation theorem

$$\nabla \tilde{V}_t(c; c) = E_t \left(\int_t^T \pi_s^{(t)} c_s ds \right) = V_t^{1-\gamma} = (1-\gamma)\tilde{V}_t.$$

where the state price deflator, conditional on information at time t , is given by $\pi_s^{(t)} = \frac{\pi_s}{\pi_t}$ for $t \leq s \leq T$. This shows that $(1-\gamma)\tilde{V}_t = \pi_t^{(t)} W_t$. Using market clearing, by Ito's lemma we deduce that $\sigma_M(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t)$, which is the important internalization of "prices". This can be "inverted", so we may express $\sigma_V(t) = (\sigma_M(t) - \rho\sigma_c(t))/(1-\rho)$ as well. These are the same results that we obtained for the non-ordinal version of recursive utility.

The conclusion is that for the recursive model, the two ordinally equivalent version produces the same risk premiums and the same real short rate.

10.5 Conclusions for the ordinally equivalent version

We formulate our main results of this section:

Theorem 3 *For the ordinally equivalent model in (5), in equilibrium the risk premiums of risky assets and the real interest rate are given by the same expressions as for the non-ordinal version of recursive utility.*

In this section we recover the expression for risk premium of Duffie and Epstein (1992a), which they derive using dynamic programming. They do not present the equilibrium real interest rate. Also in solving the associated Bellman equation they need to assume constant coefficients in the stochastic differential equations in order to provide explicit results, something we do not need.

Our model for the stock market is simpler than the one used by Duffie and Epstein (1992a), in that we do not require there to be a set of unspecified factors.

These authors also claim that "the unnormalized aggregator (f_1, A_1) is convenient for obtaining the desired disentangling by changing A_1 with f_1 fixed. Such a separation is much less readily described in terms of the normalized aggregator $(f_2, 0)$." Therefore it is satisfying to have the complete analysis also for the aggregator (f_1, A_1) under weaker assumptions.

We have demonstrated that, under our assumptions, these two versions have identical risk premiums and short rates.

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