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«Descartes' Parabola» and the Traditional Parabola

A Reconstruction of a Historical Method with the Help of the «Mathematica» Program

Theivendram Vigneswaran

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DECLARATION

I hereby declare that the work reported in this thesis was exclusively carried out by me under the supervision of Professor Reinhard Siegmund-Schultze. It describes the results of my own independent research except where due reference has been made in the text. No part of this thesis has been submitted earlier or concurrently for the same or any other degree.

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Signature

Date:

Certified by:

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Name and signature of the supervisor

Date:

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T.Vigneswaran

ABSTRACT

This study concentrates on Descartes' geometry, especially the Descartes' parabola and traditional parabola. Who is Descartes? René Descartes (1596-1650) was a 17th century French philosopher, mathematician and a man of science whose work, *La géométrie*, includes his application of algebra to geometry from which we now have Cartesian geometry. His work had a great influence on both mathematicians and philosophers. In mathematics Descartes chief contribution was in analytical geometry. Descartes made other known contributions to mathematics. He was the first to use the first letters of the alphabet to represent known quantities, and the last letters to represent unknown ones. Descartes also formulated a rule known as Descartes' rule of signs, for finding the positive and negative roots of an algebraic equation.

First, this study concentrates on the Descartes' studies of Pappus' problem. Also I explicitly explain how Descartes' found the traditional parabola and Descartes' parabola, and how he used the four and five lines Pappus' problems.
Secondly, this study concentrates on the Descartes' "construction" [that means geometrical solution] of equations by using Descartes' parabola and the traditional parabola. I clearly explain Descartes' construction of third and fourth degree equations by circle and traditional parabola, and the construction for fifth and sixth degree equations by using circle and Descartes' parabola. Finally, I also explain the construction of higher degree equations.

Furthermore I give three numerical examples by solving them with the mathematica program, which was designed by Stephen Wolfram.

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1. Descartes’ biography

René Descartes

Descartes was born on March 31, 1596 in La Haye, Touraine, France and died at Stockholm on February 11, 1650. Descartes' parents were Joachim Descartes and Jeanne Brochard. His mother died the year following his birth. His father was a lawyer and magistrate, which left little time for raising a family. René and his brother and sister, Pierre and Jeanne, were raised by their grandmother.

He was educated at the Jesuit college of La Flèche in Anjou. He entered the college at the age of eight years, just a few months after the opening of the college in January 1604. He stayed there until 1612, studying classics, logic and traditional Aristotelian philosophy. He also learnt mathematics from the books of Clavius. The school had made him understand how little he knew, the only subject which was satisfactory in his eyes was mathematics. Descartes spent a while in Paris, apparently keeping very much to himself, and then he studied at the University of Poitiers. From 1620 to 1628 he travelled through Europe, spending time in Bohemia (1620), Hungary (1621), Germany, Holland and France (1622-23). Descartes became tired of the continual travelling and decided to settle down in Holland in 1628 and he began work on his first major treatise on physics, Le Monde, ou Traité de la Lumière. In Holland Descartes had a number of scientific friends as well as continued contact with Mersenne.

Descartes was pressed by his friends to publish his ideas and, although he was adamant in not publishing Le Monde, he wrote a treatise on science under the title ''Discours de la methode pour bien conduire sa raison et chercher la verité dans les sciences''. The treatise was published at Leiden in 1637 and Descartes wrote to Mersenne saying (J J O'Connor and E F Robertson, December 1997, see below web sources): *I have tried in my "Dioptrique" and my "Météores" to show that my Méthode is better than the vulgar, and in my "Géométrie" to have demonstrated it.* The work describes what Descartes considers is a more satisfactory means of acquiring knowledge than that presented by Aristotle's logic. Only mathematics, Descartes feels, is certain, so all must be based on mathematics. As appendices to the Discours of 1637 Descartes published *Optics, Meteorology, and Geometry*, a collection of essays. La Dioptrique is a work on optics and, although Descartes does not cite previous scientists for the ideas he puts forward, in fact there is little new. However many of Descartes' claims are not only wrong but could have easily been seen to be wrong if he had done some easy experiments. Primarily interested in mathematics, he founded *ANALYTIC GEOMETRY*, originated the *CARTESIAN COORDINATES*, and Cartesian curves. Descartes meditations on first philosophy, was published in 1641. The most comprehensive of his works, Principia Philosophiae was published in Amsterdam in 1644. In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm and he broke the habit of his lifetime of getting only up at 11 o'clock. During his lifetime, Descartes was just as famous as an original mathematician, scientist, and philosopher. Descartes is one of the most important Western philosophers of the past few centuries.

2. Introduction

In my present study, the "Descartes' parabola" is in the centre how it is defined and how it can be constructed and for which purpose it is being used. To answer these questions, I shall explain the methods of Descartes used to solve geometrical problems. Descartes published his ideas in 1637 in a treatise called *La Géométrie* (*Geometry*).

Descartes' *La Géométrie*, book I is on ''problems, the construction of which requires only straight lines and circles'', Book II is on the ''nature of curved lines''; but Descartes shows that this book was written as a necessary preparatory work to the third book, and the last, book III is on ''the construction of solid and super solid problems''. Descartes *La Géométrie* is well known as an important event in the history of mathematics. *La Géométrie* is a book, which is difficult to read. Descartes says:

''But I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind by working over it, which is in my opinion the principal benefit derived from this science. Because, I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra, who will consider carefully all that is set forth in this treatise.'' (p.10)

In the centre of this work is ''Descartes' parabola''. It is a Pappus five line locus (The word *locus* (plural *loci*) is Latin for "place"), namely a cubic curve which Newton called "trident" and others (not quite clear who first) the ''Cartesian parabola''. This curve reappears frequently in *La Géométrie.* Boyer (1956) said that '' his triple interest in the curve was limited to the following three aspects: (1) deriving its equation as that of a Pappus locus; (2) showing its constructability by kinematic means; (3) using it in turn to construct the roots of equations of higher degrees''. In my work, I shall clearly explain how to use this curve in the geometrical construction of the roots of equations of fifth and sixth degrees.

2.1 A first look at Descartes' geometry

There have been many studies on Descartes' *Géométrie.* We know that Descartes' geometry contains his invention of analytic geometry. At a first look at Descartes' geometry, we may be surprised about what is not there. We do not see the analytic geometry of the straight line, or of the circle or of the conic section, we do not see Cartesian coordinates, and we do not see any curve plotted from its equation. Descartes did not use the term ''analytic geometry''. The best source for the actual contents of the *Géométrie* is the book *Géométrie* itself. In his book, Descartes does include algebra, theory of equations, classifying curves by degree, point wise construction of curves, construction of equations (eg: third, fourth, and sixth) etc.

Solution of problems in Descartes' ''La géométrie'' (1637)

Geometrical problems

Determinate:
''problems'' proper = finitely many lengths, points etc.

Geometrical classification:
plane, solid, linear

Geometry
Algebra

Equations in one unknown x,
Transformations

Algebraic classification: degree

Indeterminate:
curves, e.g. Pappus: infinitely many points

Geometrical classification:
plane, solid, linear

Geometry
Algebra

Equations in two unknowns x,y,
''Coordinate System''

Algebraic classification: degree

solution by:

algebraic manipulation (determ. and indeterm. problems)
← intersection of curves (solving determ. problems by indeterm. probl.)
→ constructing curves from points (solving indeterm. problems by determ. probl.)

[Reinhard Siegmund-Schultze (2003), p.238]

2.2 The background of Descartes' geometry

We first look at some achievements of the ancient Greeks. There are three classical problems in Greek mathematics which were extremely influential in the development of geometry. These problems were those of squaring the circle, doubling the cube and trisecting an angle. During the Greek times doubling of the cube was the most famous, and then in modern times the problem of squaring the circle became the more famous, especially among amateur mathematicians. Although it is difficult to provide an accurate date as to when the problem of trisecting an angle first appeared, we know that Hippocrates, who made the first major contribution to the problems of squaring a circle and doubling a cube, also studied the problem of trisecting an angle. The trisecting an angle was known to Hippocrates (J J O'Connor and E F Robertson, April 1999, see below web sources). I shall explain the trisecting an angle here because Descartes solved the third degree equations by using trisection method. It works as follows. Let construct a right triangle ΔABC with $\angle CAB$; and draw a line DN parallel to AB. draw ALN, intersecting BC in L and $LN = 2MN$. then the required angle is $\angle BAL$.

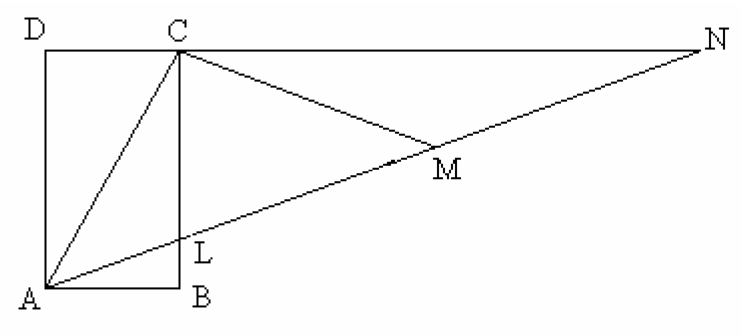


Figure 1: Trisecting an angle

Let M be the midpoint of LN so that $LM = MN = AC$. since LCN is a right angle, $LM = MN = CM$. Hence, by isosceles triangles, $\angle CAM = \angle CMA$.

Also, $\angle CMA = \angle MCN + \angle CNM$ and $\angle CAM = \angle CMA = 2\angle MNC$.

But $\angle BAL = \angle MNC$ [$AB // DN$]. So, $3\angle BAL = \angle CAB$.

Now one of the reasons why the problem of trisecting an angle seems to have attracted less in the way of reported solutions by the best ancient Greek mathematicians is that the construction above, although not possible with an unmarked straight edge and compass, is nevertheless easy to carry out in practice. Also, they solved a range of locus problems, some very complicated. To find their solutions, they too had ‘’methods’’. The construction of two mean proportionals attracted most imitation in the sixteenth century. J.M.Bos (2001, p.27) stated that ‘’ there were two related reasons fro this pre-eminence. The first was that the list of 12 different constructions of two mean proportionals that Eutocius had included in his commentary to Archimedes’ *Sphere and Cylinder* became available in print, first in works of Valla and Werner, later in editions of Archimedes’ works. A similar, though smaller set of constructions of the trisection in Pappus’ *Collection* became known only much later. Secondly, mathematicians learned and found that several problems not solvable by straight lines and circles could be reduced to the problem of two mean proportionals, whereas fewer, if any, problems were found to be reducible to trisection; so the former problem acquired a central position among problems beyond the constructional power of straight lines and circles.’’ In early modern geometers more often met solid problems reducible to two mean proportionals than problems reducible to trisection.

Given a problem, for example, consider the famous problem of doubling the cube. In modern term, the problem is, to find x such that $x^3 = 2a^3$ (given a^3). Hippocrates of Chios showed that this problem could be reduced to the problem of finding two mean proportionals between a and $2a$ (J.Grabiner (1995), p.84). That is, $a : x = x : y = y : 2a$ or $a / x = x / y = y / 2a$

The x, y will be called the two ‘’mean proportionals’’ between a and $2a$.

Then, eliminating y, we observe $x^3 = 2a^3$ as required.

If we consider the first two terms, $a : x = x : y$, we get $x^2 = ay$, which represents a parabola.

If we consider the first and last terms, $a : x = y : 2a$, we get $xy = 2a^2$, which represents a hyperbola. Thus the problem of duplicating the cube is reducible to the problem of finding the intersection of a parabola and a hyperbola. This reduction developed the Greek interest in the conic section.

Suppose we need to learn how to construct an angle bisector, and how to bisect a line segment. In figure 2, draw AD bisecting the angle A. Then the length AB = length AC and connect B and C with the line segment BC, see figure 2. Let M be the intersection of the angle bisector with the line BC. But $AB = AC$, $\angle BAM = \angle CAM$, and $AM = AM$, then the triangle $\Delta ABM \equiv$ triangle ΔACM. Thus M bisect BC.

Now, to construct the angle bisector, constructs $AB = AC$, construct the line BC, bisect it at M, and connect the points A and M. AM bisects the angle A.

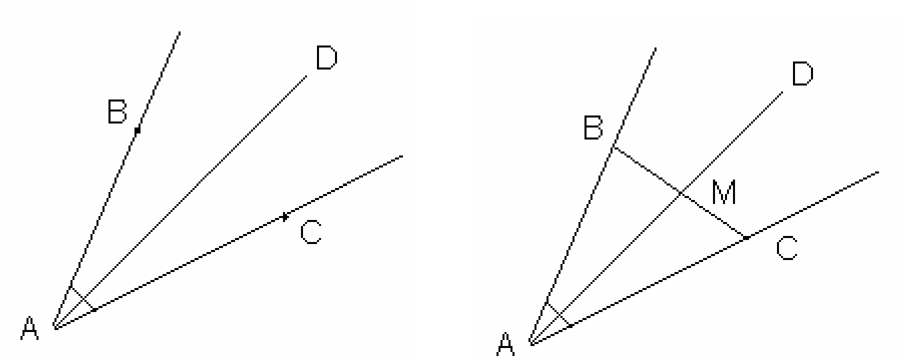


Figure 2: Bisect an angle and line

The Greek ‘’analysis’’ works like this. Descartes did not invent these methods. Descartes ideas on problem solving, moreover, have other antecedents besides the Greek mathematical tradition (J.Grabiner (1995), p.85).

Further, in excellent work of Greek mathematics (Euclid, Archimedes and Apollonius), there were two sorts of geometrical propositions: theorems and problems. Theorems had to be proved; problems had to be constructed (J.M.Bos, 1984, p332). Descartes extended these earlier ideas in an unprecedented way.

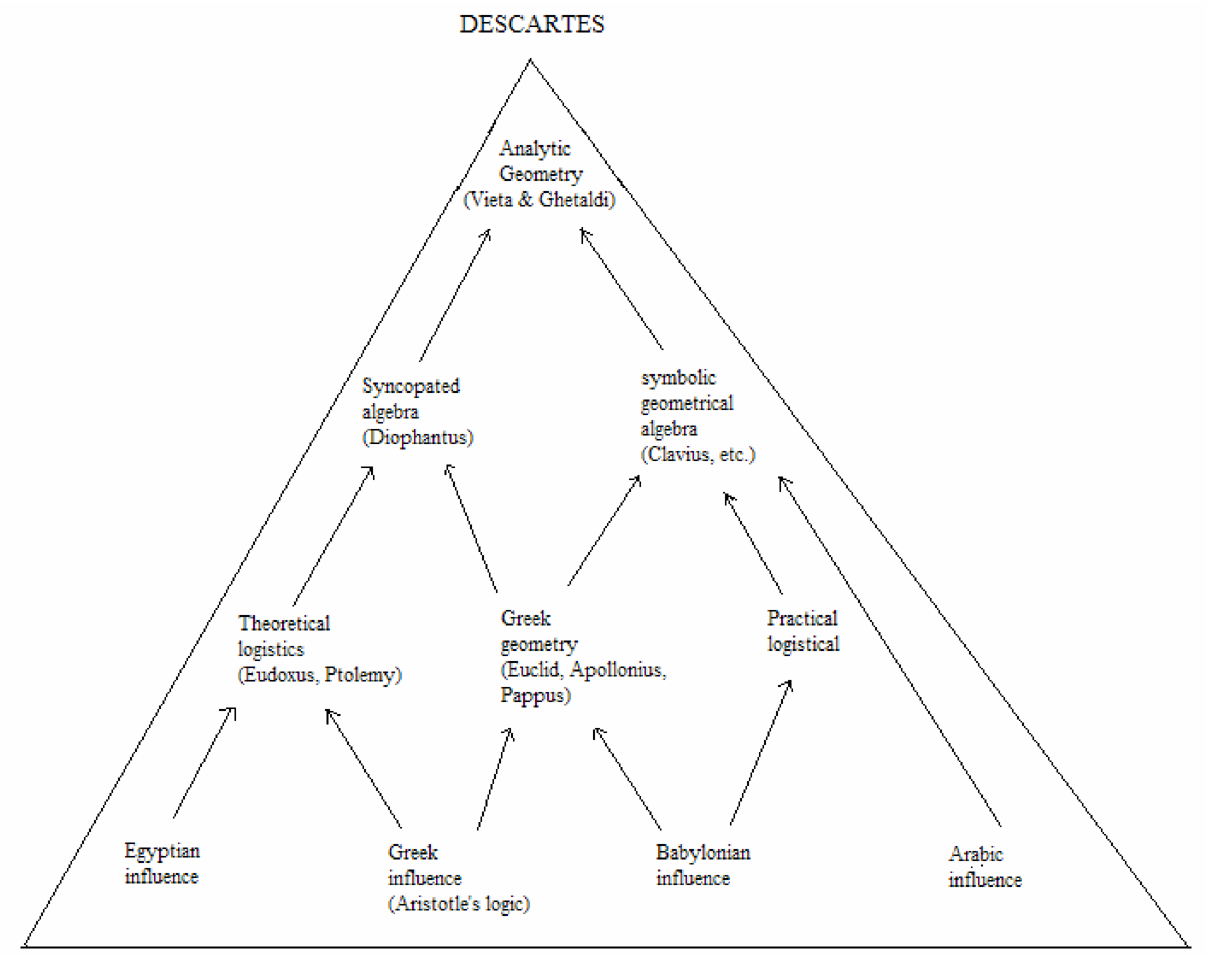


Figure 3: Ancient mathematical developments (E.G.Forbes, 1977, p.148)

3. Descartes' parabola and the traditional parabola

3.1 Descartes and curves

In the beginning section of Book II of his *Géométrie*, Descartes famously introduces his criteria for accepting curves into geometry. He claims to borrow a classification proposed by the ancients. The ancients considered three classes of geometric problems, which they called plane, solid, and linear.

Geometric problems

Plane	Solid	Linear
[problems which require only circles and straight lines for their construction]	[which can be solved by using conic section]	[more complex alg. curves]
↓	↓	↓
First degree (+ circle which is second degree)	Second degree	High degree

Descartes provided such delimitation in terms of the curves used in the procedures. The curves that he allowed in geometry are now called ''algebraic curves'', the others "transcendental". Descartes used the term of ''mechanical curve'' that cannot be expressed by an algebraic equation. But Leibniz and others called them "transcendental". Descartes distinction between ''geometrical'' and ''mechanical'' curves provided a great issue in seventeenth century mathematics.
Bos says:

> *''Descartes introduced a sharp distinction between admissible and inadmissible curves. The first he called ''geometrical'' the other ''mechanical''. The ''geometrical'' curves are what we now call algebraic curves (although Descartes did not explicitly say as much in the Géométrie, this can be inferred from what he did state); the ''mechanical'' curves are those which are now termed transcendental curves''* (H.J.M. Bos, 1981, p.297)

Curves

Geometrical — Mechanical [Some sort of instrument has to be used]

Descartes groups “geometrical” curves into distinct classes. For instance, curves of “the first and simplest class” (ie, the circle, parabola, hyperbola, and ellipse) are described by a first or second degree polynomial equations and can be pointwise constructed by straight lines and circles. (cf. Bos 1996, Mahoney 1969) Following the methods delimited by Descartes, only the so-called “algebraic” curves, those with a corresponding closed polynomial, are to be included in the sphere of geometry. In contrast, so-called “transcendental” curves are rendered geometrically unintelligible.

3.2 Pointwise construction of curves

Descartes solved the Pappus problem by constructing arbitrarily many points on the locus. In the first book of *Géométrie* he did not say (Pappus problem) whether this pointwise construction could be considered as a construction of the locus as a curve. Descartes did not stop after giving the pointwise construction; He also gave the name of the locus curve (parabola, ellipse, and hyperbola etc.) and giving its basic parameters. However, he returned to pointwise constructions of curves and wrote that in the second book. In certain cases, the pointwise construction curves should be accepted in geometry (J.M.Bos, (1981), p.315). I shall clearly explain the Pappus problem in my present work below.

3.4 Descartes and parabola

In the geometry of plane curves, the term parabola is often used to denote the curves given by the general equation $a^m x^n = y^{m+n}$, thus $ax = y^2$ is the quadratic or Apollonian parabola; $a^2 x = y^3$ is the cubic parabola, $a^3 x = y^4$ is the biquadratic parabola; semiparabolas have the general equation $ax^{n-1} = y^n$, thus $ax^2 = y^3$ is the semi cubical parabola and $ax^3 = y^4$ the semibiquadratic parabola. These curves were investigated by René Descartes, Sir Isaac Newton, Colin Maclaurin and others (http://www.1911encyclopedia.org/Parabola).The

Cartesian parabola is a cubic curve which is also known as the *Descartes' parabola* of Descartes on account of its form. Its equation of the form is $y^3 - 2ay^2 - a^2y + 2a^3 = axy$. I shall explain the Descartes' parabola (cubic curve) and traditional parabola below and also discuss it how it was constructed by Descartes.

3.5 The Pappus problem

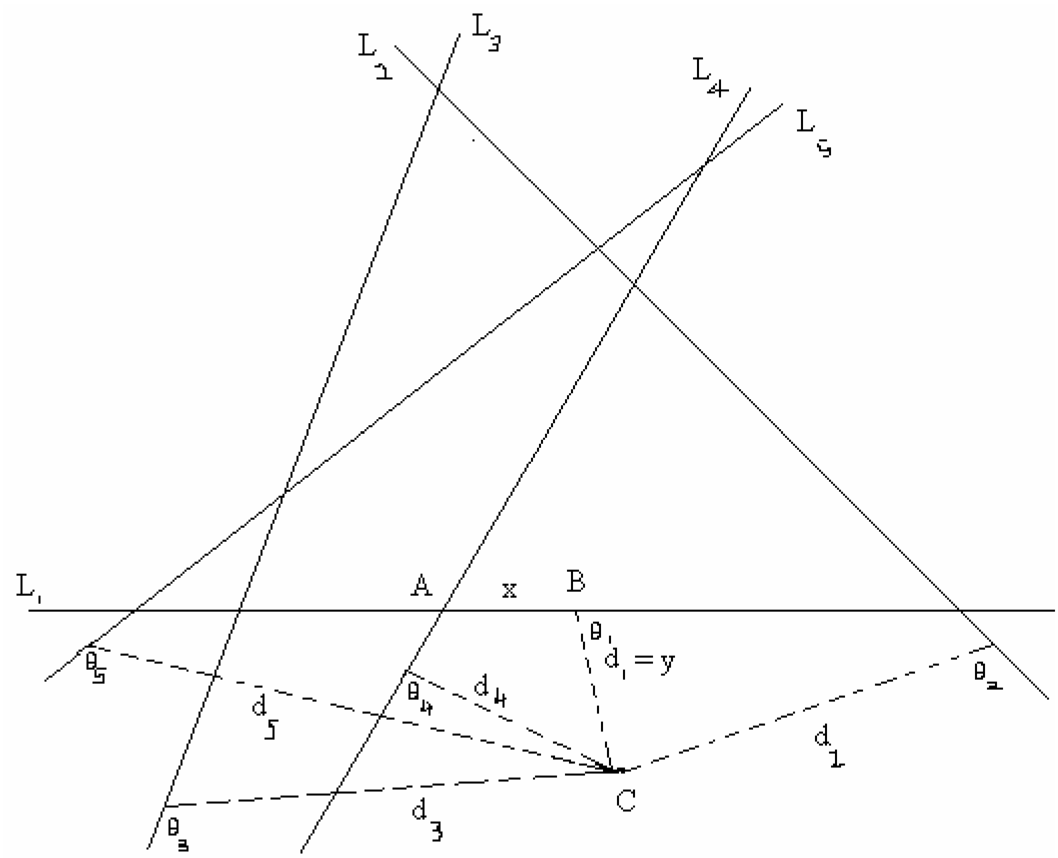


Figure 4: Pappus' problem

I shall explain the Pappus problem and Descartes' solution of this problem. Given the straight lines $L_i, i = 1,2,.........,n$ in the plane, and angles θ_i be fixed angles, d_i, $i = 1,2,............,n$ denote the distances of the line from an arbitrary point C in the plane to L_i. The given line segment a and K be a given constant ratio (involving the distances d_i and depending on the number of lines L_i). The related ratios (by us written as products) are:

For three lines: $d_1^2 = Kd_2 d_3$

For four lines: $d_1 d_2 = Kd_3 d_4$

For five lines: $d_1 d_2 d_3 = Kad_4 d_5$

For six lines: $d_1 d_2 d_3 = Kd_4 d_5 d_6$

- -
- -
- -
- -
- -
- -

For an even number $2n$ of lines:

$$d_1 d_2 \ldots\ldots\ldots\ldots d_n = Kd_{n+1} d_{n+2} \ldots\ldots\ldots\ldots\ldots\ldots d_{2n}$$

$$\prod_{i=1}^{n} d_i = K \prod_{i=n+1}^{2n} d_i$$

For an odd number $2n+1$ of lines:

$$d_1 d_2 \ldots\ldots\ldots\ldots d_{n+1} = Kad_{n+2} d_{n+3} \ldots\ldots\ldots\ldots\ldots\ldots d_{2n+1}$$

$$\prod_{i=1}^{n+1} d_i = Ka \prod_{i=n+2}^{2n+1} d_i$$

I use modern notation for this problem. Descartes did not use indices and stated the coefficients clearly with respect to a figure. In his formulation Descartes surely meant the generality that modern notation can express.

Pappus provides the problem for three and four lines as well as its generalization to more lines. Pappus' problem was a locus problem (*Locus*: the set of points satisfying a particular condition, often forming a *curve* of some sort.). In each case there are infinitely many points which satisfy the given condition; these points form a locus in the plane; this locus is generally a curve. Pappus also says that for three and four lines the locus is a conic section. Further, for more than four lines nothing is known about the form of the locus.

3.6 The general solution

At the end of the first book of his *Géométrie,* Descartes gives the general solution of the problem (p. 309-314). His idea is given below. He assumed that

$$d_1 = y$$

ie) Descartes singles out L_1 as the reference line (in our terminology the x-axis of a coordinate system).

Descartes takes x to be the distance along L_1 (line) from a fixed point to the intersection of d_1 with L_1 at point B. He then shows by geometrical arguments that all d_i can be expressed linearly in x and y:

$$d_i = a_i x + b_i y + c_i$$

The coefficients a_i, b_i, and c_i are constants belonging to the line segments L_i and the given angle θ_i and also the a_i, b_i, and c_i are known. Descartes also remarks that in the exceptional case when all lines are parallel, x doesn't occur in the expressions for the d_i.

The constancy of the given ratio K can be expressed as an equation:

For an even number $2n$ of lines:

$$y(a_2x + b_2y + c_2).........(a_nx + b_ny + c_n) = K(a_{n+1}x + b_{n+1}y + c_{n+1}).........(a_{2n}x + b_{2n}y + c_{2n})$$

ie) $\prod_{i=1}^{n}(a_i x + b_i y + c_i) = K\prod_{i=n+1}^{2n}(a_i x + b_i y + c_i)$

For an odd number $2n+1$ of lines:

$$y(a_2x + b_2y + c_2).......(a_{n+1}x + b_{n+1}y + c_{n+1}) = Ka(a_{n+2}x + b_{n+2}y + c_{n+2})......(a_{2n+1}x + b_{2n+1}y + c_{2n+1})$$

ie) $\prod_{i=1}^{n+1}(a_i x + b_i y + c_i) = Ka\prod_{i=n+2}^{2n+1}(a_i x + b_i y + c_i)$

Where $d_1 = a_1x + b_1y + c_1 = y$ and $a_1 = c_1 = 0$ and $b_1 = 1$.

The degrees of these equations depend on the number of lines. If there are three or four lines this results in the second degree of these equations. I shall explain this case below ($y^2 = 4x$). Descartes did not explicitly discuss the degrees of these equations, but he was aware of them. The interpretation of the original problem would require the d_i to remain positive. But Descartes did not discuss the negative values. Further Descartes discussed only one curve, but the original interpretation would lead to a locus consisting of two curves. The equation should be

For an even number $2n$ of lines:

$$|y||a_2x+b_2y+c_2|........|a_nx+b_ny+c_n| = |K||a_{n+1}x+b_{n+1}y+c_{n+1}|........|a_{2n}x+b_{2n}y+c_{2n}|$$

or $y(a_2x+b_2y+c_2)........(a_nx+b_ny+c_n) = \pm K(a_{n+1}x+b_{n+1}y+c_{n+1})........(a_{2n}x+b_{2n}y+c_{2n})$

Similarly, for an even number $2n+1$ of lines:

$$y(a_2x+b_2y+c_2)......(a_{n+1}x+b_{n+1}y+c_{n+1}) = \pm Ka(a_{n+2}x+b_{n+2}y+c_{n+2}).....(a_{2n+1}x+b_{2n+1}y+c_{2n+1})$$

Descartes explained the classification of curves according to the degree of their equations, in the second book (p.48). He says all geometrical curves have algebraic equations. These classifications are

First class: The curves with equations of the second degree (the circle, the parabola, the hyperbola, and the ellipse).

Second class: The curves with equations of the third and fourth degree.

Third class: The curves with equations of the fifth and sixth degree. And so on.

But Descartes said there was an exceptional case when all lines are parallel, namely, he did explicitly explain the five lines (parallel) problem in his second book (p.83-88). I shall clearly explain below.

3.7 Pappus problem in four lines

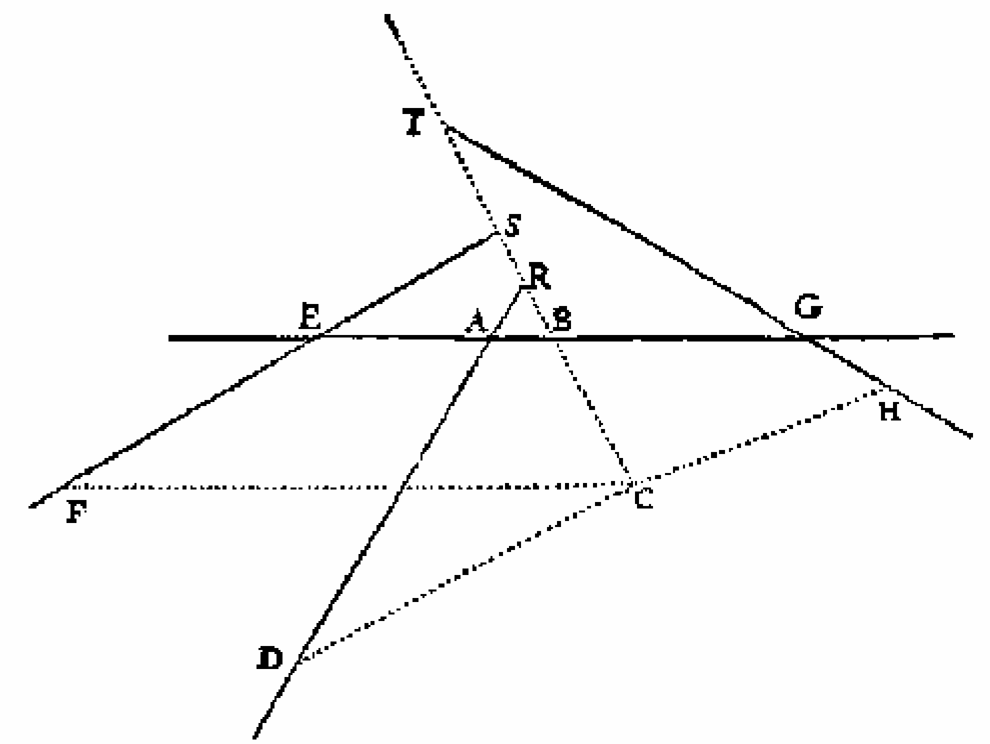


Figure 5: Pappus' problem in four lines (Géométrie, p.27/31)

Descartes explained in much more detail the four line Pappus problem in his book II. But he started to write this problem in book I and continued the work in book II. I will consider the Descartes solution of the Pappus problem for four given lines here. Descartes talked about the rectangle of two of the lines as related to the rectangle of the two other lines which of course has to be understood algebraically as the ‘’ product of distances’’.

For four lines: $d_1 d_2 = K d_3 d_4$

Descartes explained the way of getting the equation of the locus in his book I and II. I have not changed Descartes' symbols here. Descartes introduced symbols for the unknowns and the given parameters in this Pappus problem as given below.

Assume that $AB = x$ and $BC = y$

But these lines are not parallel. The given lines intersect AB in the points A, G, E and intersect BC in the points R, T, S. We can see this figure above.

Now, we can consider the triangle ΔABR, where all angles are known. So, the ratios of the sides are known.

Suppose that $\frac{AB}{BR} = \frac{z}{b}$ then we get, $BR = \frac{bx}{z}$ [$\therefore AB = x$]

Since $CR = y + \frac{bx}{z}$ when B lies between C and R

[Other cases: $CR = y - \frac{bx}{z}$ when R lies between B and C, $CR = -y + \frac{bx}{z}$ when C lies between B and R]

Now, we can consider the triangle $\Delta\ CRD$, where all angles are known. So, the ratios of the sides are known.

Let $\frac{CR}{CD} = \frac{z}{c}$ then we get, $CD = \frac{cyz + bcx}{z^2}$ [$\therefore CR = y + \frac{bx}{z}$]

Also, the distance is known from A to E. That is $AE = k$.

So, then $BE = x + k$

[Other cases: $BE = -x + k$ when B lies between A and E, and $BE = x - k$ when E lies between A and B]

Again we can consider the triangle $\Delta\ BSE$, where all angles are known. So, the ratios of the sides are known.

So, the ratio is $\frac{BE}{BS} = \frac{z}{d}$ then we get, $BS = \frac{kd + xd}{z}$

And also $CS = y + BS$ $[\therefore BC = y]$

$$\therefore CS = \frac{yz + kd + xd}{z}$$

[Other cases: $CS = \frac{yz - kd - xd}{z}$ when C lies between B and C, and $CS = \frac{-yz + kd + xd}{z}$ when C lies between B and S]

Again we can consider the triangle ΔFCS, where all angles are known. So, the ratios of the sides are known.

Hence the ratio is $\frac{CS}{CF} = \frac{z}{e}$. Therefore, $CF = \frac{e.CS}{z}$

$$\therefore CF = \frac{ezy + ekd + ekx}{z^3} \quad [\therefore CS = \frac{yz + kd + xd}{z}]$$

And assume $AG = l$ then $BG = l - x$

Again we consider the triangle ΔBGT, the ratio is $\frac{BG}{BT} = \frac{z}{f}$

This implies, $BT = \frac{fl - fx}{z}$ and $CT = \frac{zy + fl - fx}{z}$

Also the triangle ΔCHT, then the ratio is $\frac{CT}{CH} = \frac{z}{g}$ and

Hence, $CH = \frac{gzy + gfl - gfx}{z^2}$ $[\therefore CT = \frac{zy + fl - fx}{z}]$

We know that z was known. Descartes found the line segments $CB, CF, CD,$ and CH. These are

$$CB = y$$

$$CF = \frac{ezy + ekd + ekx}{z^3}$$

$$CD = \frac{cyz + bcx}{z^2}$$

$$CH = \frac{gzy + gfl - gfx}{z^2}$$

Descartes provided this information in his book I and he gave the solution of Pappus four lines problem. That is, to find all points C in the plane with $d_1 d_2 = K d_3 d_4$

Descartes considered the given ratio K to be equal to 1. But he did not write any further words on this point in his book. So, the product of BC and CF is equal to the product of CD and CH.

That is, $BC.CF = CD.CH$

This implies, $y.\left[\frac{ezy+ekd+exd}{z^2}\right]=\left[\frac{czy+bcx}{z^2}\right].\left[\frac{gzy+gfl-gfx}{z^2}\right]$

$$y^2[ez^3-cgz^2]=y[cfglz-ekdz^2]-xy[edz^2+cfgz-bcgz]+bcfglx-bcfgx^2$$

The equation is

$$y^2=\frac{y[cfglz-ekdz^2]-xy[edz^2+cfgz-bcgz]+bcfglx-bcfgx^2}{[ez-cg]z^2}$$

And he assumed that $ez > cg$, then $ez^3-cgz^2>0$ and its square root is therefore real. Also Descartes assumed that $2m=\frac{cf\lg z-ekdz^2}{ez^3-cgz^2}$ and $\frac{2n}{z}=\frac{edz^2+cfgz-bcgz}{ez^3-cgz^2}$

Then we get, $y^2=2my-\frac{2n}{z}xy+\frac{bcfglx-bcfgx^2}{ez^3-cgz^2}$

The roots of this quadratic equation is

$$y=m-\frac{nx}{z}+\sqrt{m^2-\frac{2mnx}{z}+\frac{n^2x^2}{z^2}+\frac{bcfglx-bcfgx^2}{ez^3-cgz^2}}$$

Again he assumed that $O=-\frac{2mn}{z}+\frac{bcfgl}{ez^3-cgz^2}$ and $\frac{p}{m}=\frac{n^2}{z^2}-\frac{bcfg}{ez^3-cgz^2}$

Then we get the root of this quadratic equation:

$$y=m-\frac{nx}{z}+\sqrt{m^2+Ox+\frac{p}{m}x^2}$$

That is, $BC=m-\frac{nx}{z}+\sqrt{m^2+Ox+\frac{p}{m}x^2}$

I shall explain the conic section (see appendix, p.77) here by above equation. These cases are

If $\frac{p}{m}x^2 = 0$ then the conic section is a parabola.

If $\frac{p}{m}x^2 > 0$ then the conic section is a hyperbola.

If $\frac{p}{m}x^2 < 0$ then the conic section is an ellipse.

I shall continue to explain the case of parabola below. Because this is my present work so I omit the other cases. Also Descartes explicitly explained the hyperbola and ellipse cases in his geometry book II.

3.8 Traditional Parabola

I draw this figure but I have no change the Descartes symbols here.

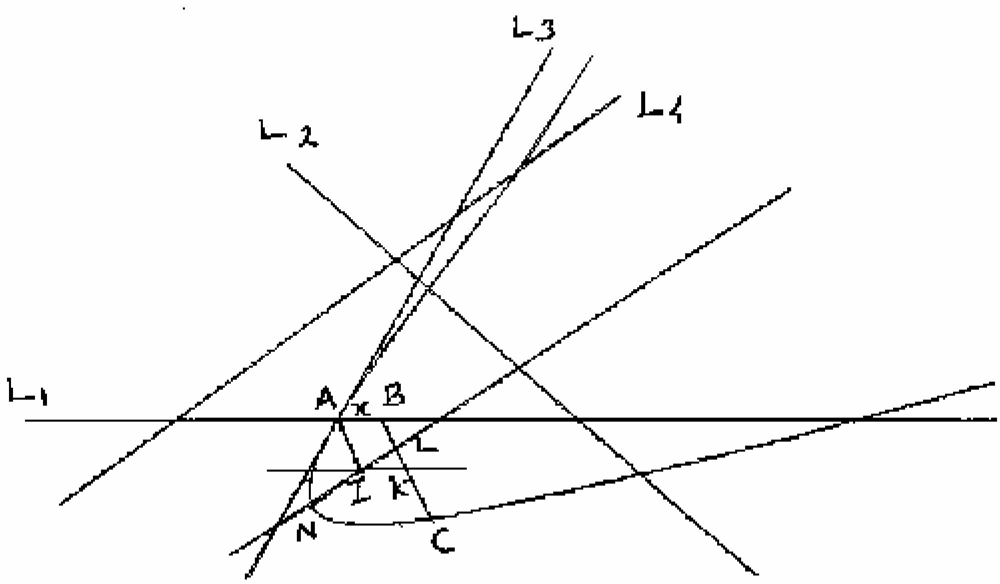


Figure 6: Pappus' problem in four lines: especially for the construction in the case of parabola

NI is the principal axis in the parabola and also I is the focus. Then draw a line IK equal and parallel to AB, and intersecting at a BC point K. Also he took a line segment $BK = m$ (for BC contain $+m$, if this were $-m$ then the line IK on the other side of AB, and if $m = 0$ then we cannot draw IK at all) and $IK = AB = x$.

Further, Descartes did not clearly draw a point at I in his figure. But I draw clearly in my figure above. Then he assumed that $\frac{IK}{KL} = \frac{z}{n}$

That is, $KL = \frac{n}{z}x \quad [\therefore IK = x]$

Similarly, we know that the ratio $\frac{KL}{IL} = \frac{n}{a}$ this implies $IL = \frac{a}{z}x$.

If $-\frac{n}{z}x$, he took the point K between L and C and if $+\frac{n}{z}x$, the point L between K and C, and if $\frac{n}{z}x = 0$, we can't draw IL.

He considered BC equal to $BK - LK + LC$ on this case.

That is, $BC = BK - LK + LC$

This implies, $y = m - \frac{n}{z}x + LC$

ie) $LC = y - m + \frac{n}{z}x$

Claim: If the conic section is a parabola, the line segment is $LC = \sqrt{m^2 + Ox}$.

I shall explain this below:

If the point C is on the parabola, its latus rectum is equal to r (In a conic section, the *latus rectum* is the chord parallel to the directrix through the focus. In a parabola, the length of the latus rectum is equal to four times the focal length, i.e. the distance of the focus from the vertex. That is, "Latus rectum" is a compound of the Latin *latus*, meaning 'side,' and *rectum*, meaning 'straight') and its parabola axis on the line IL. Its vertex, N, and let $IN = \lambda$.

$$LN = LI + IN$$

$$= \frac{a}{z}x + \lambda \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

The condition of parabola $LC^2 = LN.r$

$$= \left(\frac{a}{z}x + \lambda\right).r \quad [\therefore by (i)]$$

But $LC = y - m + \frac{n}{z}x$

This implies, $\left(y - m + \frac{n}{z}x\right)^2 = \left(\frac{a}{z}x + \lambda\right).r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

If the equation of a parabola, $\left(y - m + \frac{n}{z}x\right)^2 = m^2 + Ox$ ....(iii)

Hence, $\left(\frac{a}{z}x + \lambda\right).r = m^2 + Ox \quad [\therefore by(ii) \text{ and } (iii)]$

Equating the coefficients, we get

$$\frac{a}{z}r = O \text{ This implies } r = \frac{Oz}{a}$$

And $\qquad \lambda r = m^2$

$$\therefore \lambda = \frac{am^2}{Oz}$$

That is, $IN = \frac{am^2}{Oz}$ and latus rectum is equal to $\frac{Oz}{a}$.

Hence the equation of the parabola is $y = m - \frac{nx}{z} + \sqrt{m^2 + Ox}$.

Descartes also explained the plane loci are degenerate cases of solid loci in his book II (If the line is straight or circular, it's called a plane locus and if it is a parabola, a hyperbola, or an ellipse, it's called a solid locus). Moreover, the different kinds of solid loci represented by the equation

$$y = \pm m^2 \pm \frac{n}{z}x \pm \frac{n^2}{x} \pm \sqrt{\pm m^2 \pm Ox \pm \frac{p}{m}x^2}$$

by Rabuel. Descartes omitted the case in that neither x^2 nor y^2 but only xy occurs, and the case in that a constant term occurs (p.79).

If $\frac{n^2}{x}$ is not present, there the quantity under are several cases here.

(i) If the radical sign is zero or a perfect square, then equation is a *straight line.*

(ii) If this quantity is not a perfect square and if $\frac{p}{m}x^2 = 0$, then the equation is a *parabola.*

(iii) If this quantity is not a perfect square and if $\frac{p}{m}x^2 < 0$, then the equation is a *circle* or an *ellipse*.

(iv) If $\frac{p}{m}x^2 > 0$, then the equation is a *hyperbola*.

If all the terms of the right hand side is zero except $\frac{n^2}{x}$, then the equation is a hyperbola referred to its asymptotes.

3.9 Numerical example:

1] Suppose all the given quantities expressed numerically, as

$EA = 3, AG = 5$

$AB = BR, BS = BE, GB = BT$ and

$CD = \frac{3}{2}CR, CF = 2CS, CH = \frac{2}{3}CT$ and also the angle $\angle ABR = 60^\circ$

We are referring to Descartes' picture (see figure 5) and that all quantities where B, C and S come in are variable.

Then all these quantities must be known if the problem is to be entirely determined.

Now, let $AB = x$, and $CB = y$.

Then $CR = CB + BR = y + x$ ($AB = BR = x$)

But $CD = \frac{3}{2}CR = \frac{3}{2}(y + x)$ …………..(i)

And $BS = BE = EA + AB = 3 + x$

$CS = CB + BS = y + x + 3$

But $CF = 2CS = 2(y + x + 3)$ …………….(ii)

Then $BG = AG - AB = (5 - x)$ and also $GB = BT = (5 - x)$

$CT = CB + BT = y - x + 5$

So, $CH = \frac{2}{3}CT = \frac{2}{3}(y - x + 5)$ ……………(iii)

Descartes gives the property of four lines is:

$CB.CF = CD.CH$ (Assume $K = 1$)

$y[2(y + x + 3)] = [\frac{3}{2}(y + x)][\frac{2}{3}(y - x + 5)]$

$2y^2 + 2xy + 6y = 5y - xy + y^2 + xy - x^2 + 5x$

$y^2 + 2xy + y + x^2 - 5x = 0$;

$y^2 + y(1 + 2x) + (x^2 - 5x) = 0$;

So, $y = -\frac{1}{2} - x \pm \sqrt{\frac{1}{4} + 6x}$

Hence $BK = m = -\frac{1}{2}, \frac{n}{z} = 1$ and $O = 6.$ but the line IK on the other side of AB ($m = -\frac{1}{2}$) and $IK = AB = x$, $KL = x$, $\angle IKL = \angle ABR = 60^\circ$, $IL = x$, and the quantity represented by z is 1, we get $a = 1$ and $r = \frac{Oz}{a} = 6$ (See figure 6). It follows that the curve NC is a *parabola* and its latus rectum is equal to 6.

2] An example treated in modern terms, we consider the parabola $y^2 = 4x$ in rectangular (x, y) – coordinates which results from a four lines Pappus problem in the following way:

I consider the four straight lines: $y = x, y = -x, x = 0$ and $x = 4$

The angles are 45 degrees for the two first distances, and 90 degrees for the other two distances on the right hand side of the equation. That the parabola is indeed the solution to the four lines Pappus problem follows from the equation (ie. $K = 1$)

$$d_1 d_2 = d_3 d_4$$

$$(y - x)(y + x) = x(4 - x)$$

This implies, we get $y^2 = 4x$

The example should be supported by a picture of the parabola together with the four lines in the plane.

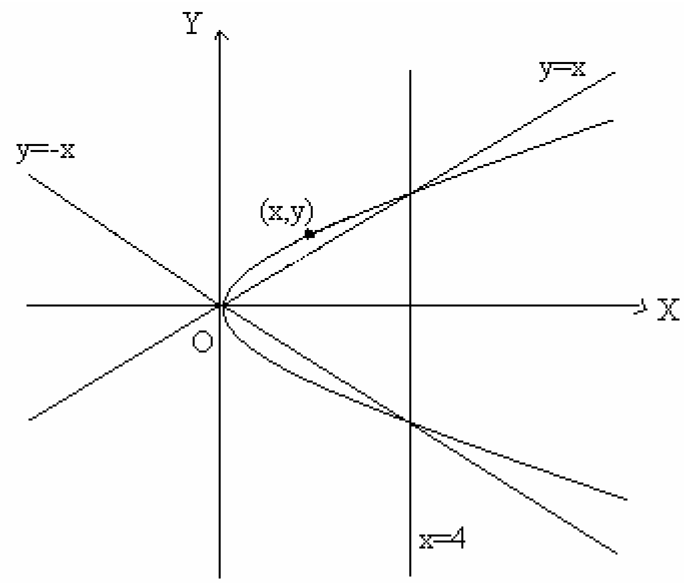


Figure 7: Numerical example for the Pappus' problem in four lines

Descartes did not discuss the fact that as a result of this rewriting the d_i may have negative values, whereas the obvious interpretation of the original problem would require the d_i to remain positive. The effect of this is that Descartes found only one curve as locus, while the original interpretation would lead to a locus consisting of two curves. (H.J.M Bos, 1981, p.300)

The four line problem (ie. $K=1$), Descartes worked out

$$y(a_1x+b_1y+c_1)=(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)$$

And found one conic section as the locus. But if the d_i were taken to be positive, the equation would become (modern term)

$$|y||(a_1x+b_1y+c_1|=|a_2x+b_2y+c_2||a_3x+b_3y+c_3|$$

or
$$y(a_1x+b_1y+c_1)=\pm(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)$$

That is two conics. We could consider the above example of this case. (k=1)

$$|x||x-4|=|x-y||x+y|$$

or
$$x(x-4)=\pm(x-y)(x+y)$$

$$y^2=4x \text{ or } 2x^2-4x=y^2$$

These are two conics. But $2x^2-4x=y^2$, this is a hyperbola.

I plot this hyperbola by using mathematica program (see further details at page 35).

Plot [$\sqrt{2x^2-4x}$, { x,–6,6}];

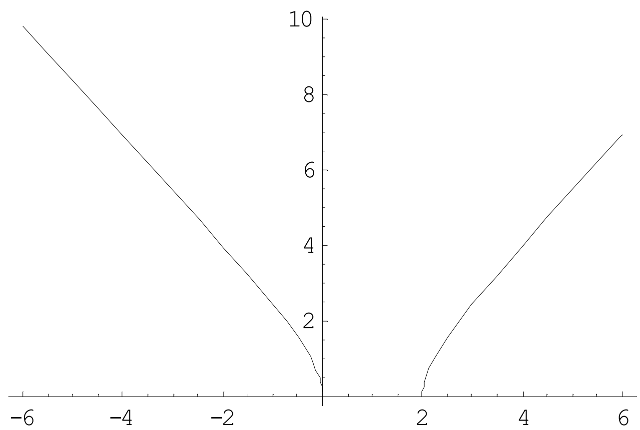


Figure 8: Example for the Pappus' problem in four lines: hyperbola

3.10 Pappus five line problem

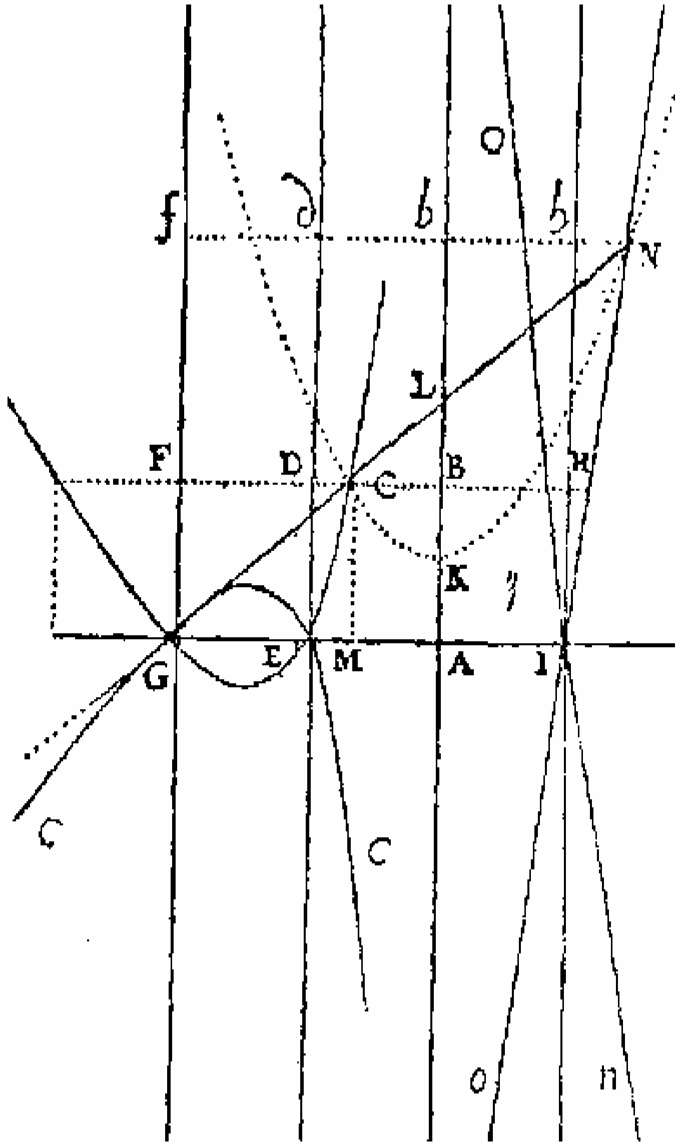


Figure 9: Pappus' problem in five lines

In the second book (p.84) Descartes showed the solution of a special simple case of the five line Pappus problem. I would like to consider this problem here.

Method 1:

Let the given lines be $AB, IH, ED, GF,$ and GA respectively. In the situation where $AB, IH, ED,$ and GF are parallel and equal distance and GA is perpendicular to the other lines. Let the required point C lies between AB and DE, the distance $CB, CF, CD, CH,$ and CM

are perpendicular to the given lines. We can see the figure (p.82/6) above. Descartes gives the property of five lines

$$CF.CD.CH = CB.CM.a$$

Descartes takes $CF = (2a - y)$, $CD = (a - y)$ and $CH = (a + y)$ and he gave the equation of the curve is:

$$(2a - y)(a - y)(a + y) = axy$$

or $$y^3 - 2ay - a^2 y + 2a^3 = axy$$

This is the Cartesian Parabola.

Where $CB = y$, $CM = x$ and $GE = EA = AI = a$. The ruler GL is moving around G.

Method 2: *[p.84]*

I shall explain Descartes' second method of the required cubic curve. Descartes consider a parabola CKN with vertical axis KL to move up and down the straight line AB (That is, the parabola CKN is moving vertically along its axis AB) and the principal parameter equal to a (that is, the parameter corresponding to the axis of the given parabola) of the parabola $y^2 = ax$. The ruler GL is moving around G. Also the straight line GL can intersect of the lines GF and GI while L moves along AB. He takes KL equal to a of the vertical axis of parabola CKN.

We consider $CB = MA = y$, $CM = AB = x$, and $GA = 2a$.

Also we consider the triangles ΔCMG and ΔCBL are similar.

So, this implies $\frac{GM}{CM} = \frac{CB}{BL}$

$$\Rightarrow \frac{(2a - y)}{x} = \frac{y}{BL}$$

ie) $$BL = \frac{xy}{(2a - y)}$$

But $KL = a$ and this implies

$$BK = KL - BL$$

$$BK = a - \frac{xy}{(2a - y)}$$

or $$BK = \frac{2a^2 - ay - xy}{2a - y}$$

Then BK is a segment of the axis of the parabola $y^2 = ax$.

Since $\frac{BK}{BC} = \frac{BC}{a}$, a is equal to the principal parameter.

This implies, $BC^2 = a.BK$

$$y^2 = a.\frac{(2a^2 - ay - xy)}{(2a - y)}$$

$$2ay^2 - y^3 = 2a^3 - 2a^2 y - axy$$

That is, $y^3 - 2ay^2 - 2a^2 y + 2a^3 = axy$

The combined motion the points C of intersection of the parabola and the straight line move over the plane; they trace a new curve CEG; this curve is the required five line locus. The point C can be taken on the curve GEC which is a branch of the '' Cartesian parabola''. NIo and nIO are intersection of the line GL with the other branch of the original parabola KN. Descartes also discussed the opposite direction of the parabola.

This curve played an important role in his theory of geometrical construction and this third degree curve which later known as the name ''Cartesian parabola''. Descartes also did not explain how he had found the way. Descartes explained how the ''Cartesian parabola'' can be used for finding the roots of sixth and fifth degree equation in his *Géométrie* book III. He also discussed how the ''Cartesian parabola'' was traced by the combined motion of a ruler and a traditional parabola. I shall explain more details in my present chapter below. Descartes explained in the one particular case (that is, the property of five lines is: $CF.CD.CH = CB.CM.a$) of the Pappus five line problem in his book II. Descartes did not explain the further choice of five line problem in his *Géométrie* book II.

If we consider four given parallel lines $AB, IH, ED,$ and GF, and one perpendicular cutting line GA, Descartes gives the distance property of five lines is

$$ad_i d_j = d_k d_m d_n$$

If we can consider the five different distances, so the possible permutation (rearrangement of distances) is the numbers 1,2,3,4 and 5. Descartes found the cubic curve in the particular case $CF.CD.CH = CB.CM.a$ (see above). J.M.Bos (2001, p.330) analysed the possible type of the five line problem and also he divided into two cases.

These are

$$ayd_i = d_j d_k d_m \quad \ldots\ldots\ldots\text{I}$$

$$ad_i d_j = yd_k d_m \quad \ldots\ldots\ldots\text{II}$$

Bos denoted d_5 by y. Descartes case belongs to the case I. (If you want to further information, see J.M.Bos book on page 330/31).

Descartes then stated that the given lines $GF, DE, AB,$ and HI are parallels non-equidistant and the line GA is not perpendicular to the others (see below). In this case, he says, the required point C will not always lie on curve of the same nature and this may even meet be the case when the given lines are not two parallel.

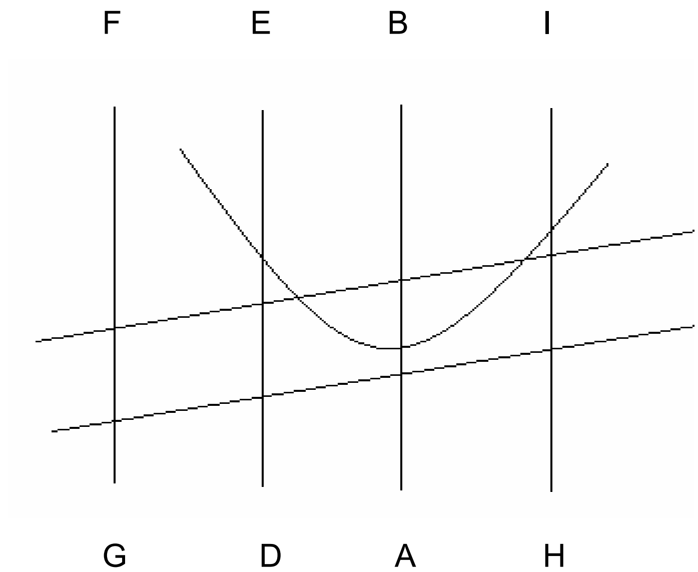


Figure 10: Pappus' problem in five lines: non-equidistant parallel lines and the fifth line is not perpendicular

3.11 Another five line locus

Descartes explained the "parallelepiped" (that means the product of three distances) of three lines drawn through the point C for the one cutting line and any two of the parallel lines is equal to the parallelepiped of two lines drawn through point C to meet the other two parallels and another given line (a). He says that the required point lies on a curve of different nature.

The distance property of five lines is (case II)

$$ad_i d_j = yd_k d_m$$

He wrote (p.88):

''In this case the required point lies on a curve of different nature, namely, a curve such that, all other ordinates to its axis being equal to the ordinates of a conic section, the segments of the axis between the vertex and the ordinates bear the same ratio to a certain given line as this line bears to the segments of the axis of the conic section having equal ordinates. I cannot say that this curve is less simple than the preceding; which nevertheless I believed should be taken as the first, since its description and calculation are somehow easier.''

But Rabuel gave the general equation of this curve: $axy - xy^2 + 2a^2x = a^2y - ay^2$ (p.88)

3.12 Numerical example:

I would like to give a good numerical example (J.M Bos (1981), p.316) for a five line problem here. If we take the origin in the centre of the figure,

$$d_1.d_2.d_3 = d_4.d_5.a$$

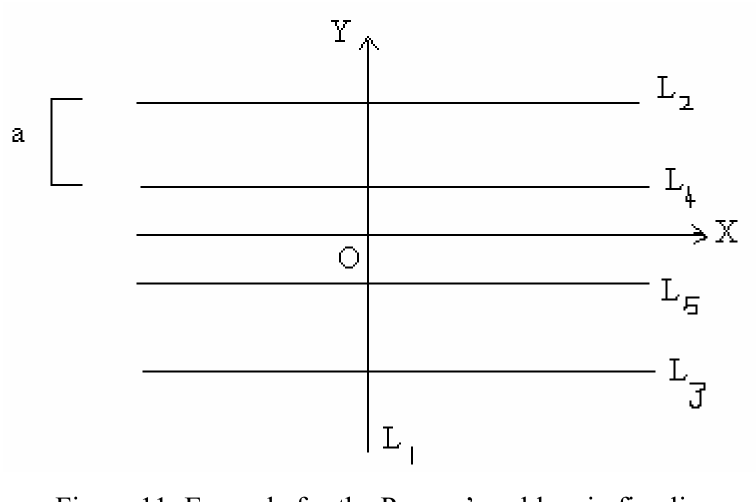


Figure 11: Example for the Pappus' problem in five lines

Leads to $x\left(y+\frac{3a}{2}\right)\left(y-\frac{3a}{2}\right)=\left(y+\frac{a}{2}\right)\left(y-\frac{a}{2}\right)a$

or $4xy^2 - 4ay^2 - 9a^2x + a^3 = 0$

as the equation for the required curve. Taking $w = \frac{1}{2}a^{-1}(y^2 - \frac{9}{4}a^2)$

We find $w : a = a : (x - a)$

If we now take the ''vertex'' in Descartes text to be the point $V(x = a, y = 0)$, and draw the parabola.

$$2aw = y^2 - \frac{9}{4}a^2$$

with w taken along the X-axis from V, then the required curve and the parabola are related in such a way that for points (x, y) and (z, y) on either curve with equal ordinates y, the abscissa $(x - a)$ and w (taken from V) satisfy

$$w : a = a : (x - a)$$

This corresponds to what Descartes says, but the he does not specify that in this case the conic section is a parabola.

I plot this parabola by using mathematica program (see further details at page 35) and also I consider the constant value $a = 1$. That is, $w = \frac{1}{2}y^2 - \frac{9}{8}$

Plot $[\frac{1}{2}y^2 - \frac{9}{8}, \{y, -6, 6\}]$;

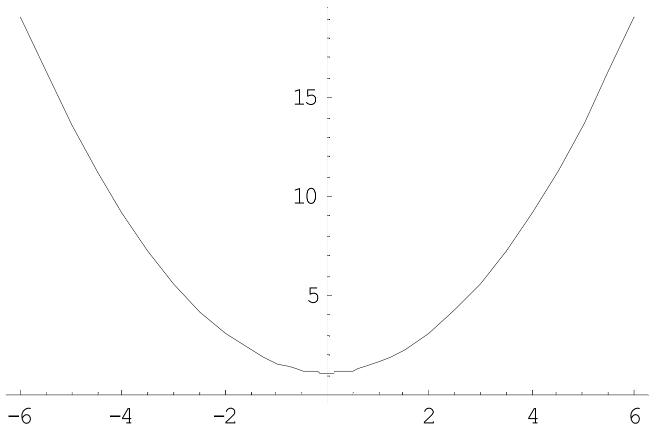


Figure 12: Example for the Pappus' problem in five lines: parabola

4. Descartes' construction of equations by using Descartes' parabola and the traditional parabola

The constructions of equations of degree three to six are discussed in the last book of the Géométrie, and they form the conclusive result of the treatise. I shall discuss these constructions below. If the degree was one or two, the roots could be constructed by straight lines and circles. Descartes called these problems ''plane''. If the degree was three or four, its roots could not be constructed by circles and straight lines. Descartes found the roots of any equation of third or fourth degree could be constructed by the intersection of a circle and a parabola. Also he called the problem ''solid'', because only conics (= "solid curves") are involved. Now, I can consider the equation of fifth or sixth degree. If the equation was of degree five or six Descartes called the problem '' super solid''. Descartes gave a new curve namely the ''Cartesian parabola''. Descartes showed that this curve was really the solution of the five line Pappus problem and he provided the equation of the curve, in his second book [p.83-84]. This curve became later known as the ''Cartesian parabola'' and other names for the curve are '' trident'' (Newton) and ''parabolic conchoid''. It is, however, no classical parabola, but a curve of third degree:

$$y^3 - 2ay^2 - a^2 y + 2a^3 = axy$$

I shall explain this curve in my chapter below. In the third book Descartes explained how this curve can be used for finding the roots of fifth and sixth degree equations; he stated there in more detail how the curve was traced by the motion of a ruler and a parabola [p.220]. This case is complicated but basically correct. In the book III on the last page Descartes showed the constructing equations of higher degree than six. Descartes did not give any work of this case. He wrote:

> *''……..but one degree more complex by cutting a circle by a curve but one degree higher than the parabola, it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum….''* [p.240]

In this present chapter I shall explain how he had constructed 5^{th} and 6^{th} degree equations by using Descartes' ''parabola''. I shall discuss some examples here, from the third book of *Géométrie*. Descartes used the different geometrical techniques in this construction of equations.

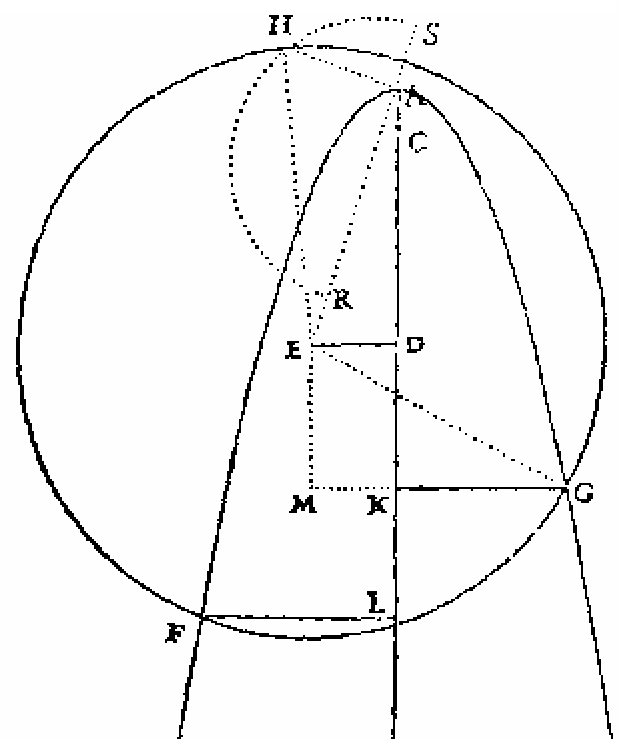
4.1 Descartes' construction of fourth degree equations by circle and the traditional parabola

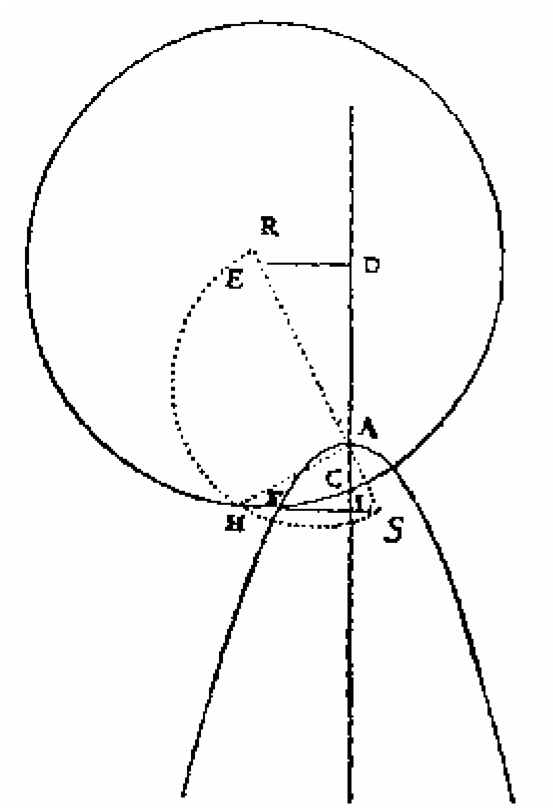
Descartes provided his constructions of equations of third and fourth degree (that means he found the roots by his geometrical method) in the last book of the geometry. Also he assumed that the cubic term of the equation was omitted. He wrote the equation as

$$x^4 = \pm apx^2 \pm a^2qx \pm a^3r \text{ ……………………..(*)} \qquad \text{[p.195]}$$

Where $p, q,$ and r are positive.

If $r = 0$, the fourth degree equation (*) reduces to a cubic equation. In this case of the cubic equation the intersection at a point A (the vertex point A is on the circle) corresponds to the root $x = 0$ (we can see below fig. p.206). I shall prove in one example below. We can see below figures for all cases of fourth degree equations (p.194/7/8).





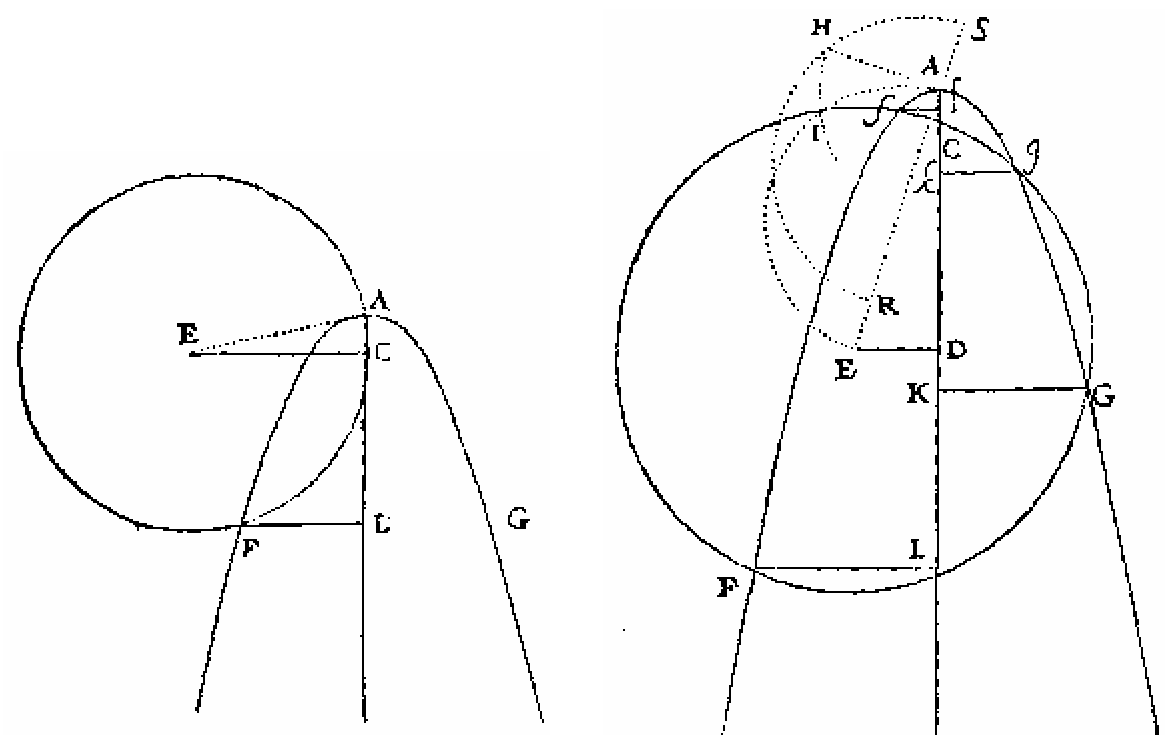


Figure 13: The construction of third and fourth degree equations

We can see here how to construct the fourth degree equations by a circle and a parabola. Let AL be the vertical axis of the parabola and vertex A as the highest point on the parabola. We assume the point A is an origin for the (x,y)- coordinate system along the horizontal direction GK and the vertical axis of parabola, respectively. I would like to consider the latus rectum a of this given parabola $x^2 = ay$. Let E be the centre and d the radius of the circle. Assume that the line $GK = x$ then $AK = \frac{x^2}{a}$. Since, GK is the mean proportional between AK and the latus rectum a of the parabola. Find D on the vertical axis such that $AD = \frac{1}{2}(p + a)$ and draw $DE = \frac{1}{2}q$ perpendicular from D.

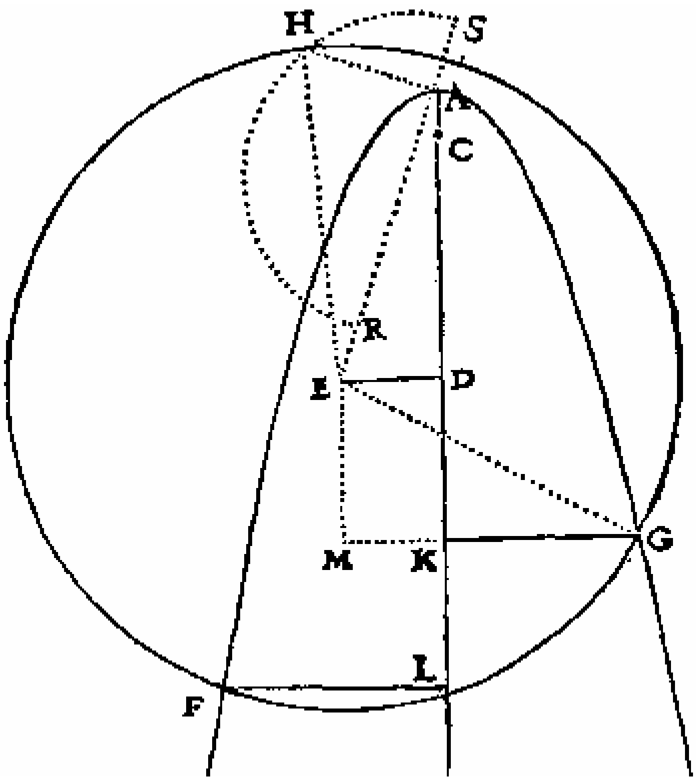


Figure 14: The construction of fourth degree equations

We have

$$EG^2 = EM^2 + MG^2$$

$$= [\frac{x^2}{a} - \frac{1}{2}(p+a)]^2 + (x + \frac{1}{2}q)^2$$

$$= [\frac{x^4}{a^2} - \frac{x^2}{a}(p+a) + \frac{1}{4}(p^2 + 2ap + a^2) + [x^2 + qx + \frac{1}{4}q^2]$$

$$a^2d^2 = x^4 - a(p+a)x^2 + a^2x^2 + a^2qx + \frac{1}{4}a^2(p^2 + q^2 + 2ap + a^4)$$

$$x^4 - apx^2 + a^2qx + a^2[\frac{1}{4}(p+a)^2 + \frac{1}{4}q^2 - d^2] = 0$$

Descartes constructed the equation of fourth degree as

$$x^4 - apx^2 + a^2qx - a^3r = 0 \quad \text{or} \quad x^4 = apx^2 - a^2qx + a^3r$$

Where $-ar = [\frac{1}{4}(p+a)^2 + \frac{1}{4}q^2 - d^2]$ or $d^2 = [\frac{1}{4}(p+a)^2 + \frac{1}{4}q^2 + ar]$

Descartes gave on page 203 a proof (unit case) of the radius of the circle FGH.

Assume that $AD = \frac{1}{2}(p+a)$, $DE = \frac{1}{2}q$ and ΔEDA is a right triangle.

$$AE^2 = DE^2 + DA^2$$

$$= \frac{1}{4}q^2 + \frac{1}{4}(p+a)^2 \text{ .... (i)}$$

Then, consider the circle SHR. Assume that $AS = a$ (we can note that AS is equal to latus rectum a) and $AR = r$.

Since AH is the mean proportional between AS and AR.

ie) $AH^2 = AS.AR$

$$= ar \text{ ....(ii)}$$

Also, since ΔHAE is a right triangle.

$$EH^2 = AE^2 + HA^2$$

$$EH^2 = \frac{1}{4}q^2 + \frac{1}{4}(p+a)^2 + ar \quad (\therefore \text{by (i) and (ii)})$$

ie) $d^2 = \frac{1}{4}(p+a)^2 + \frac{1}{4}q^2 + ar$

If a is used as unit then the equation as

$$x^4 - px^2 + qx - r = 0 \qquad \text{or} \qquad x^4 = px^2 - qx + r$$

Descartes did prove for one of his cases (latus rectum equal to1) distinctions and left the other cases to the reader. The circle can cut or touch the parabola in maximum four points. I would like to write the Descartes own words here (p.200).

> *'' Now the circle FG can cut or touch the parabola in 1, 2, 3, or 4 points; and if perpendiculars are drawn from these points upon the axis they will represent all the roots of the equation, both true and false''*

The line segments FL, GK, gk and fl [see figure 13] are the roots of the equation. The intersection point F on the left of the axis gives the ''true'' [ie. positive] root; any on the other side correspond to ''false'' [ie. negative] roots. Descartes observed if the circle and parabola may not intersect or touch at any point, there is no root but they are all imaginary.

But Descartes stated that If the value q is positive then the line segment FL a true root, a point E (centre of the circle, see figure 14) on the same side of the axis of the parabola; while the others, as the line segment GK, will be false roots. Otherwise, if the value q is negative then the true roots (gk and GK) will be those on the opposite side and the false or negative roots (FL) will be those on the same side as E. Descartes here mentioned the negative roots. Also if the circle did not cut or touch the parabola at any point, then all the roots are imaginary (p.200).

If the value q is zero (he did not say), the given equation will be reduced to a quadratic equation.

ie) $x^4 = px^2 + r$

Let $y = x^2$ (say)

$y^2 = py + r$ ………………… (i)

Descartes did not comment on this quadratic equation. But he had already explained the solution of quadratic equations (i) in his first book of *Géométrie* (p.13).

If $r = 0$, then the fourth degree equation reduced to cubic equation (p.196).

ie) $x^4 = apx^2 + a^2qx$ or $x^3 = apx + a^2q$

And the radius of the circle is equal to $d = \sqrt{\frac{1}{2}(p+a)^2 + \frac{1}{4}q^2}$

But, AE is equal to $\sqrt{\frac{1}{2}(p+a)^2 + \frac{1}{4}q^2}$

ie) $AE = d$

So, the circle passes through at a point A. The circle intersects the parabola in the points F, G, g and A. The lines segments FL, GK and gk are the roots of the cubic equation.

I explain in modern terms the solution of the fourth degree equation here.

Assume that $GK = x$, and $AK = y$ then $y = x^2$ because G is on the parabola.

We take the parabola $y = x^2$ with vertical axis and latus rectum is equal to 1.

But G is also on the circle and the centre E of the coordinates are $\left\{\frac{1}{2}q, \frac{1}{2}(p+1)\right\}$

The equation of the circle is $\left[x - \frac{1}{2}q\right]^2 + \left[y - \frac{1}{2}(p+1)\right]^2 = d^2$

$$x^2 + y^2 - qx - (p+1)y + \left[\frac{1}{4}q^2 + \frac{1}{4}(p+1)^2\right] = d^2$$

The circle $x^2 + y^2 - (p+1)y - qx - r = 0$, where $r = d^2 - \left[\frac{1}{4}q^2 + \frac{1}{4}(p+1)^2\right]$

Then find from the equation as,

$$x^2 + x^4 - (p+1)x^2 + qx - r = 0$$

$$\text{ie)}\ x^4 - px^2 + qx - r = 0$$

{The circle of the general equation is $x^2 + y^2 + 2gx + 2fy + c = 0$

But the centre of the coordinates is $(-g,-f)$

The parabola has equation $y = x^2$. Assume the centre E of the circle, its coordinates (a,b) and its radius is d.

The equation is $(x-a)^2 + (y-b)^2 = d^2$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - d^2 = 0$$

Both curves are intersecting. This implies

$$x^2 + x^4 - 2ax - 2bx^2 + a^2 + b^2 - d^2 = 0$$

$$x^4 = (2b-1)x^2 + 2ax + (d^2 - a^2 - b^2)$$

Hence, we take $(2b-1) = \pm p$, $2a = \pm q$, and $(d^2 - a^2 - b^2) = \pm r$

$$b = \frac{1}{2}(\pm p - 1), a = \frac{1}{2}(\pm q), \text{ and } d^2 = \pm r + a^2 + b^2$$

This implies, $x^4 = \pm px^2 \pm qx \pm r$ }

This method is easy to understand for people today. Descartes gave only for one of his case distinctions (namely, $+p, -q,$ and $+r$) and left the other cases to the reader.

4.1.1 Numerical example:

Mathematica is a computer program designed by Stephen Wolfram (a former physicist) used in scientific computing, mathematics, economics, medicine, and many other fields. Mathematica is a computer program for doing mathematics. It is used for instruction, research, writing, and others. It is possible for both numeric and symbolic work, it contains functions which allow a computer to perform a wide range of mathematical calculations from basic algebra and geometry though the calculus of variations and number theory. More information is on the site <www.wolfram.com>.

Weaknesses: steep learning curve, an interface that is difficult to use from the command line and rather complex installation procedures.

How to use this document:

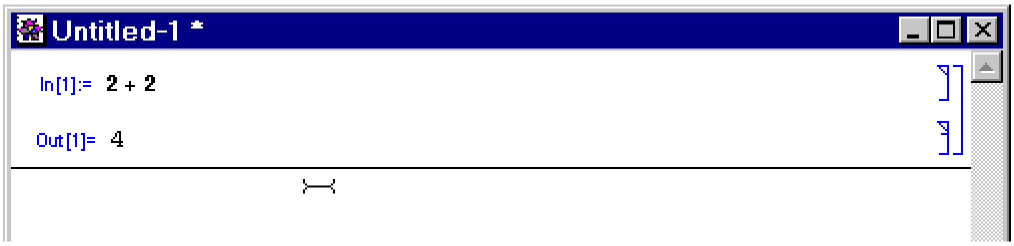
This document is intended for new users of Mathematica. No earlier math software experience is assumed, though we do point out differences between the major packages along the way. I think this operating system is easy to understand for everybody. Since Mathematica is quite visually oriented, we will be using it on a computer with a windowing system (such as Windows, Macintosh, or X-windows).

One example, we could do simple arithmetic calculation with Mathematica. If we wanted to add 2 and 2, we type the input 2 + 2 and hit Return.

In[1]:= 2 + 2

Out[1]= 4

We enter information and commands into the notebook window, and the output (if any) is displayed there



We save our work, choose File->Save As... or File->Save.

Complex numbers in mathematica:

Mathematica uses the letter ⅈ to represent the square root of -1.

Type Sqrt[-1] or $\sqrt{-1}$ and we will get the answer ⅈ

We can use ⅈ in expressions: the complex number a+bi is represented as a+bⅈ in mathematica.

Mathematica uses the function *Conjugate* to take the complex conjugate of a number. One example, the conjugate of a complex number is that number with the sign of the imaginary part reversed

ie) Conjugate (a+bⅈ)=(a-bⅈ)

Mathematica commands in conjugate:

In[1]:=a=1+2I

Conjugate[a]

Then we can ‘’ ENTER’’ and we will get the answer

Out[1]= 1+2 i

Out[2]= 1-2 i

Help system

Mathematica has an excellent help system. To get general help, choose Help->Help Browser. Browse among the topics listed.

The Mathematica Book: - Mathematica comes with an excellent resource. See also the Getting Started section, which contains several excellent tutorials (Further information, see appendix).

I would like to give a numerical example here and I shall construct the fourth degree equation by intersection of a circle and a parabola. Also I shall use the mathematica program and find the roots too. The standard form of the fourth degree equation is $x^4 = \pm px^2 \pm qx \pm r$ ($a = 1$).

I consider the fourth degree equation: $x^4 = 7x^2 - 2x + 8$

Where $p = 7, q = 2, r = 8$ and I consider the value $a = 1$. So, the equation of the parabola is $x^2 = y$.

Now I calculate the values: $AD = \frac{1}{2}(p+a) = \frac{1}{2}7 + \frac{1}{2} = 4$ and $DE = \frac{1}{2}q = \frac{1}{2}2 = 1$

Then I consider the right triangle is ΔEDA.

$$AE^2 = AD^2 + DE^2 = 16 + 1$$

$$AE = \sqrt{17}$$

Also, $HA = \sqrt{r} = \sqrt{8}$

Since ΔHAE is a right triangle.

$$EH^2 = AE^2 + HA^2 = 17 + 8$$

$$EH = 5$$

We assume the point A is an origin for the (x,y)- coordinate system along the horizontal direction GK and the vertical axis of parabola, respectively.

Let $GK = x$, $AK = y$ and the point E is (-1,4).

Then the equation of the circle is $(x+1)^2 + (y-4)^2 = 5^2$

$$\text{ie. } x^2 + y^2 + 2x - 8y - 8 = 0$$

Now I want to solve this equation. So, I choose these two curves, circle: $x^2 + y^2 + 2x - 8y - 8 = 0$ and parabola: $x^2 = y$

I consider the circle and parabola:

Circle: $x^2 + y^2 + 2x - 8y - 8 = 0$ ……………..(a) and

Parabola: $x^2 = y$ ……(b)

First, I would like to plot the two graphs. So, I use the mathematica program commands given below.

```
g1=Graphics[Circle[{-1, 4}, 5];
g2=Plot[x^2, {x, -10, 10}, DisplayFunction→Identity];
Show[g1,g2, AspectRatio→Automatic, PlotRange→{-15,15},
        Axes→True, DisplayFunction→$DisplayFunction];
```

I plot the circle and parabola on the same axis.

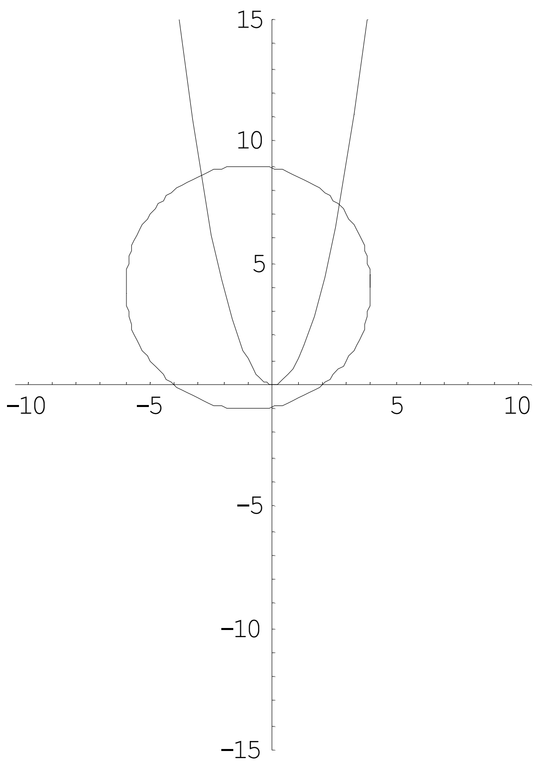


Figure 15: Numerical example for the construction of fourth degree equation

The circle $x^2 + y^2 + 2x - 8y - 8 = 0$ intersects the parabola $x^2 = y$ at two real points. Now we solve find the intersection points. We obtain a numerical approximation given below. I use the mathematica program command also given below.

Nsolve[$y == x^2 \&\& x^2 + y^2 + 2x - 8y - 8 == 0$]

{{y→8.61065,x→-2.93439},{y→7.35066,x→2.71121},{y→-0.980657+0.222411 ⅈ,x→0.111591 +0.996549 ⅈ},{y→-0.980657-0.222411 ⅈ,x→0.111591 -0.996549 ⅈ}}

Now, I shall construct the fourth degree equation by intersection of a circle ($x^2 + y^2 + 2x - 8y - 8 = 0$) and a parabola ($x^2 = y$).

We consider the two equations and substitute (b) in (a), we get the fourth degree equation

$$x^2 + (x^2)^2 + 2x - 8(x^2) - 8 = 0$$

$$x^2 + x^4 + 2x - 8x^2 - 8 = 0$$

ie) $x^4 = 7x^2 - 2x + 8,$ where $p = 7, q = 2,$ and $r = 8$

Then I shall find the roots of fourth degree equation here. So, I use the mathematica command and we get the result below. We can see the two real roots in the picture and the two complex roots found in mathematica here.

Nsolve[$x^4 - 7x^2 + 2x - 8 == 0$]

{{x→-2.93439}, {x→2.71121}, {x→0.111591 +0.996549 ⅈ}, {x→0.111591 -0.996549 ⅈ}}

Then I would like to plot the fourth degree curve $y = x^4 - 7x^2 + 2x - 8$, which gives the roots to the equation on the interval $-5 \le x \le 5$ given below. We can observe the two real roots and three stationary points (relative maxima or relative minima) in this figure below.

Plot[$x^4 - 7x^2 + 2x - 8$, { x ,-5,5}];

Assume that we want to plot the function $y = x^4 - 7x^2 + 2x - 8$ over the range of x values -5≤ x ≤5. [ie) $x\min \le x \le x\max$]

Plot[$x^4 - 7x^2 + 2x - 8$, { x ,-5,5}]; Then we can ‘’ ENTER’’ and we get such a plot. This ***Plot*** command can be used to plot virtually any one-dimensional function. Generally, the command takes: Plot[function, {range}]

The range contains three elements. The first, variable x (example) will be plotted on the horizontal axis, the second element is the lower limit on this variable, and the third element is upper limit on this variable (ie) $\{x, x\min, x\max\}$). The Plot command takes the square brackets, [ ]. Also note { x ,-5, 5} specifies a domain interval for x. (Further information, see appendix)

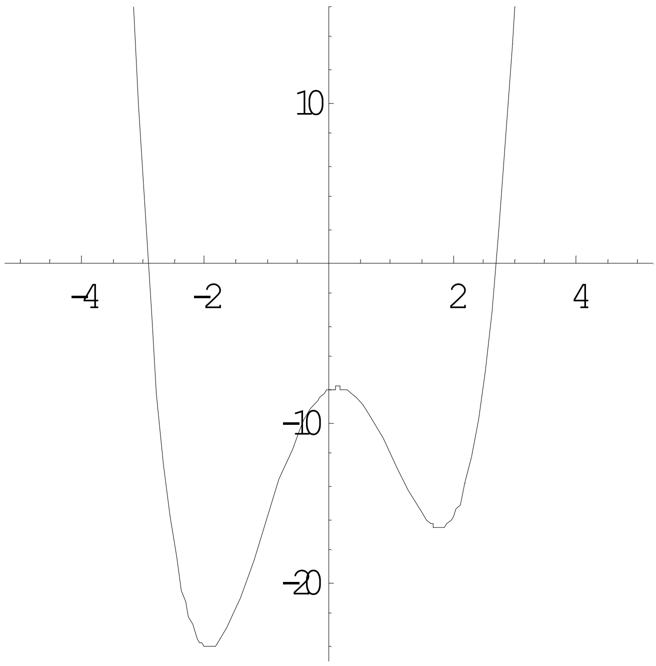


Figure 16: Fourth degree curve

4.2 Special cases for the fourth degree equation

Descartes spent much work especially on the trisection method in third degree equations and he dealt with some examples in his book III also. He discussed the two most obvious examples. These are

- Determining two mean proportionals between two line segments a and q be the way to the equation $x^3 - a^2 q = 0$
- Trisection of an angle to the equation $x^3 = 3x - q$ (NP=q be the chord subtending the given arc in a circle with radius 1)

He described clearly the outcome of the construction for two cases. I shall explicitly describe this for third degree equation below.

4.2.1 Descartes' construction of third degree equations by circle and the traditional parabola

First, Descartes explained the problem of finding two mean proportionals between the lines a and q (p.204). I shall explain the details below.

If we consider the one of the mean proportional be x, then

$$a : x = x : \frac{x^2}{a} = \frac{x^2}{a} : \frac{x^3}{a^2}$$

This implies, we get an equation and q becomes and $\frac{x^3}{a^2}$.

ie) $x^3 = a^2 q$

Then I shall explain the Descartes geometrical method. But he did not give the proof of this problem. We can consider the figure below.

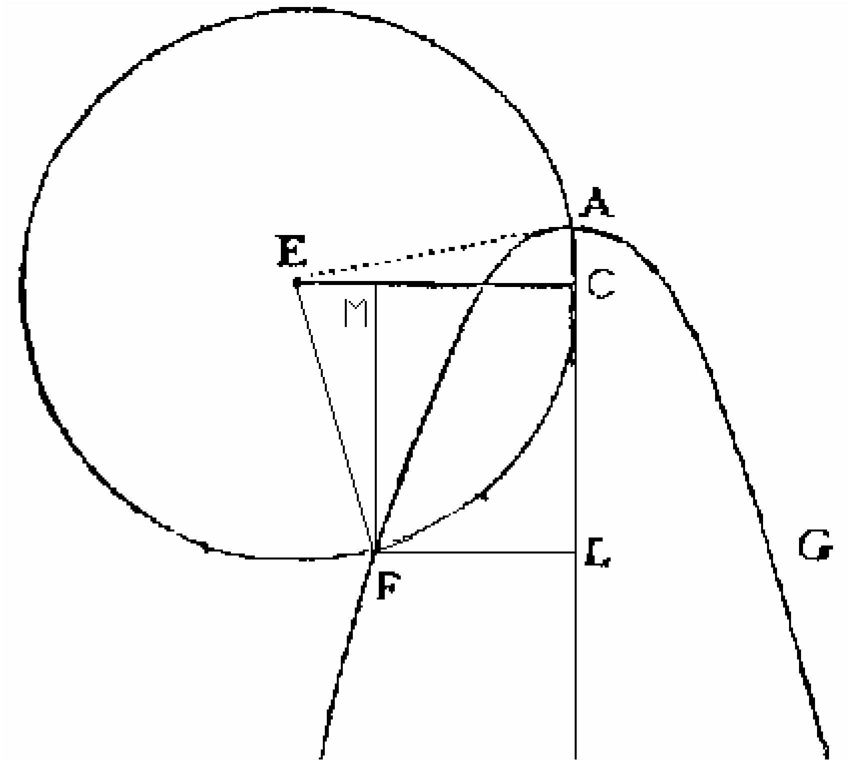


Figure 17: The construction of a third degree equation (mean proportional method)

We consider the parabola FAG with its axis AL, and the latus rectum is equal to a.

Let AC is equal to $\frac{a}{2}$ then CE is perpendicular to AC at C. ie) $CE = \frac{1}{2}q$.

Then the circle FA is passing through A and E is the centre of the circle. Then draw FM perpendicular to CE. We take the point A as an origin for the (x,y)- coordinates system along the horizontal direction FL and the vertical axis of parabola and assume $FL = x$. From the nature of the parabola, $FL^2 = a.AL$

$$\text{ie) } AL = \frac{x^2}{a}$$

Consider the right triangle ΔAEC,

$$AE^2 = AC^2 + CE^2$$

$$AE^2 = \frac{1}{4}a^2 + \frac{1}{4}q^2 \text{ .... (i)}$$

And also, $EM = EC - MC$

$$= EC - FL \quad [MC = FL]$$

$$EM^2 = (EC - FL)^2 = (\frac{1}{2}q - x)^2 \text{ .... (ii)}$$

$$FM^2 = CL^2 = (AL - AC)^2$$

$$\text{ie) } FM^2 = \left(\frac{x^2}{a} - \frac{a}{2}\right)^2 \text{ ....(iii)}$$

Then we consider the right triangle ΔEFM,

$$EF^2 = EM^2 + FM^2$$

$$EF^2 = \left(\frac{1}{2}q - x\right)^2 + \left(\frac{x^2}{a} - \frac{a}{2}\right)^2$$

$$EF^2 = \frac{1}{4}q^2 - qx + x^2 + \frac{x^4}{a^2} - x^2 + \frac{1}{4}a^2$$

But $EF = AE$ [$\because$ The radius of the circle]

This implies, $\frac{1}{4}q^2 - qx + x^2 + \frac{x^4}{a^2} - x^2 + \frac{1}{4}a^2 = \frac{1}{4}a^2 + \frac{1}{4}q^2$

$$\Rightarrow \frac{x^4}{a^2} - qx = 0$$

ie) $x^3 = a^2 q$.

Secondly, Descartes explicitly explained that the solution of third degree equations could be reduced to either finding two mean proportionals or to a trisection method and I shall clearly explain that how such construction can be found as follows here. He also gives a clear example in his *Géométrie* book III. We can consider the figure below [from pages 206/13]. Descartes wrote the third degree equation as

$$x^3 = \pm apx \pm a^2 q \text{ ....(**)} \qquad \text{[p.195]}$$

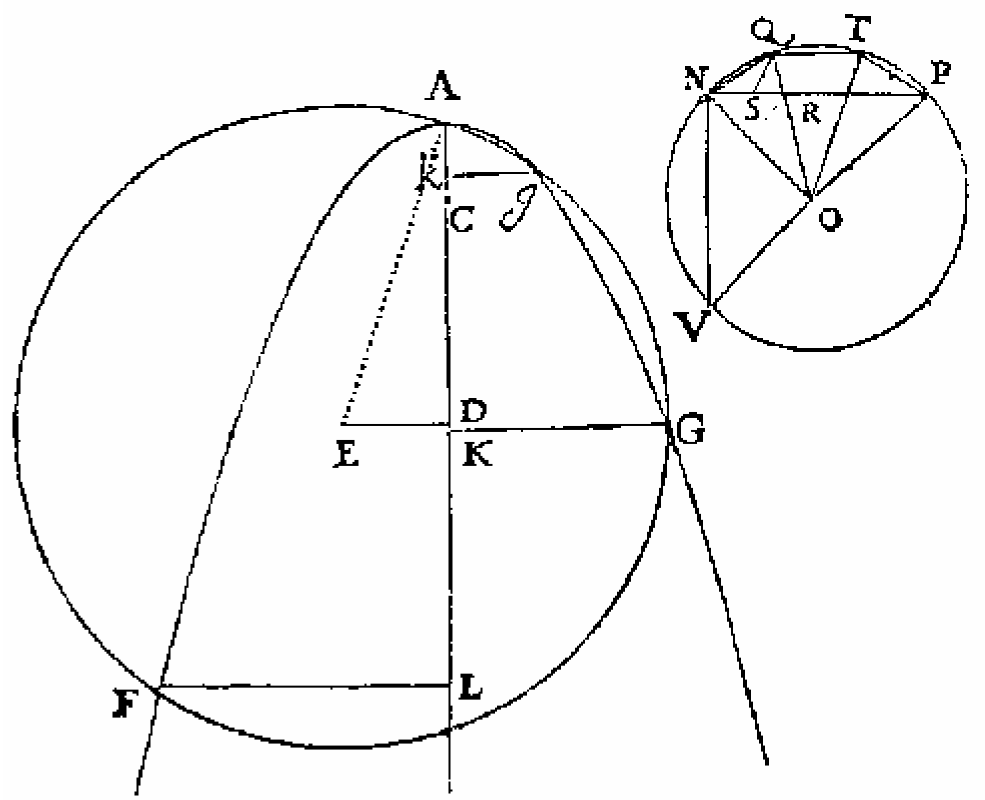


Figure 18: The construction of a third degree equation (trisection method)

Consider the circle of radius $NO = a$ (Say) and centre at O the circular arc $NQTP$, into three equal parts (ie. Draw the chord NQ, QT and TP of the three equal parts of $\angle NOP$). Let $NP = \dfrac{3q}{p}$ be the chord subtending the given arc and $NQ = x$ be the chord subtending one-third of arc. Drawing QS parallel to TO, its intersection with NP is S. Let OT cut NP at M.

$\angle NOQ$ is measured by arc NQ;

$\angle QNS$ is measured by 1/2 arc QP or arc NQ;

$\angle SQR = \angle QOT$ is measured by arc QT or NQ;

We can see that the triangles ONQ, NQR and QRS are similar.

An angle $\angle NOP$ trisected, three equal parts

$\angle OQN = \angle NQR = \angle QSR = \theta$

The triangle ΔOQN,

$NO = QO$ (Radius of circle)

So, the angles $\angle OQN = \angle ONQ = \phi$

The triangle ΔNQR,

$NQ = NR$ and the angle $\angle NQR = \angle NRQ = \phi$

And $QS \parallel OT$, then

$$\angle SQR = \angle QOT = \theta$$

So, consider the triangle ΔQSR,

$$\angle SQR = \theta \text{ and } \angle SRQ = \phi$$

$$\angle QSR = \angle QRS = \phi$$

$$QS = QR$$

ie.) All triangles $\Delta ONQ, \Delta NQR$ and ΔQRS is isosceles.

It is clear that is $NO : NQ = NQ : QR = QR : RS$

$$a : x = x : \frac{1}{a}x^2 = \frac{1}{a}x^2 : \frac{1}{a^2}x^3$$

We have,

$$\begin{aligned} NP &= 2NR + MR \\ &= 2NQ + MR \\ &= 2NQ + MS - RS \\ &= 2NQ + QT - RS \\ &= 3NQ - RS \end{aligned}$$

$$\frac{3q}{p} = 3x - \frac{1}{a^2}x^3$$

$$q = px - \frac{p}{3a^2}x^2$$

If $\frac{p}{3a^2} = 1$ then $a^2 = \frac{p}{3}$

$$a = \sqrt{\frac{p}{3}}$$

But a is a radius of the circle. ie) $NO = a = \sqrt{\frac{p}{3}}$

I can write the general equation: $x^3 = px - q$

If the circle of radius a is used as unit circle, then $p = 3$.

Descartes also explained the different cases of third degree equations. These are

$x^3 = px - q \qquad x^3 = px + q \qquad x^3 = -px + q$

We can observe that he did give the first proof of construction of third degree equation for the unity case (latus rectum is equal to 1). Also he stated the general cases, but Descartes did not give a general proof. Descartes omitted the equation $x^3 = -px - q$ because he assumed that at least one solution was positive (J.M.Bos, p.377).

I use the method and modern terms here.

I can consider the general cubic equation $x^3 + a_1x^2 + a_2x + a_3 = 0$ and the substitution $x \to \left(x - \frac{a_1}{3}\right)$, this implies $\left(x - \frac{a_1}{3}\right)^3 + a_1\left(x - \frac{a_1}{3}\right)^2 + a_2\left(x - \frac{a_1}{3}\right) + a_3 = 0$ can be reduced to a form without quadratic term: $x^3 = -px - q$

Where $p = a_2$ and $q = \left(-\frac{1}{27}a_1^3 + \frac{1}{9}a_1^2 - \frac{1}{3}a_1a_2 + a_3\right)$

Cardano's rule gives us the root

$$x = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 - \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 - \frac{1}{27}p^3}}$$

But Descartes did not comment on this case and rule in his geometry book III. I think modern mathematics student finds it a little bit difficult to understand because our modern geometry has quite a different style now. I shall discuss its roots below in my present work.

Also, the solid problem of third degree could be reduced to (second degree) an equation of the general form

$$x^3 = \pm px \pm q \quad \ldots\ldots\ldots\ldots.(***)$$

The quadratic term of the equation has been removed before. Descartes discussed the two cases of the solution of third degree equation. But he discussed the two inequalities (J.M.Bos, p.377/8). These are (consider the one case)

i) The square of $\left(\frac{1}{2}q\right)$ is greater than the cubic of $\left(\frac{1}{3}p\right)$

ii) The square of $\left(\frac{1}{2}q\right)$ is less than the cubic of $\left(\frac{1}{3}p\right)$

I shall discuss these two cases given below.

Case i: $\frac{1}{4}q^2 > \frac{1}{27}p^3$

a) Consider the equation as $x^3 = px + q$

Descartes explained that the algebraic solution of this equation by using Cardano's formula for a cubic equation. Cardano published in his great work for the method of solving cubic equations in 1545. But Cardano attributed the formula to Scipione Ferro. His rule gives us the root

$$\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 - \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 - \frac{1}{27}p^3}}$$

Descartes observed that if the inequality $\frac{1}{4}q^2 - \frac{1}{4}p^3 > 0$ the cubic root x was real. I don't know why Descartes did not argue the equality case. But I think this was obvious for him. I can consider this equality. If the equality is $\frac{1}{4}q^2 = \frac{1}{27}p^3$ then the solution of this equation is

$$x = 2\left[\sqrt[3]{\frac{1}{2}q}\right]$$

b) Consider the equation as $x^3 = -px + q$

The solution could be expressed by Cardano's formula for a cubic equation. The rule invented Cardano attributes to one by Scipio Ferreus. At present the result is usually called Cardano's formula.

His rule gives us the root

$$\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}-\sqrt[3]{-\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}$$

I would like to say it in Descartes own words here (p.211).

> *'' It is now clear that all problems of which the equations can be reduced to either of these two forms can be constructed without the use of conic sections except to extract the cube roots of certain known quantities, which process is equivalent to finding two mean proportionals between such a quantity and unity.''*

Descartes says nothing else than Cardano's formula, but unlike Cardano, who was only interested to find a solution expressed with root signs, Descartes had still to find the geometrical point, that means the mean proportional. The latter require the use of conics, while the algebraic manipulation which leads to Cardano's formula, does not presuppose any geometrical method at all.

Case ii: $\frac{1}{4}q^2<\frac{1}{27}p^3$

This is the famous "casus irreducibilis" in which Cardano's formula did not give solutions because it involved uninterpretable square roots of negative quantities (J.M.Bos, p.377). Descartes showed that the solution of equation $x^3=px+q$ $[x^3=px-q,\ x^3=-px+q,...]$ could be reduced to a trisection. Thus Descartes reduced the solution of the third degree problem to one in which two mean proportionals, and one in which the trisection of the angle had to be found. But these two problems cannot be solved with plane methods. Descartes remarked that the circle is in its shape too simple to solve the trisection and two mean proportional problems. So, I would like to write what Descartes says here (p.219).

> *''In as much as the curvature of a circle depends only upon a simple relation between the center and all points on the circumference, the circle can only be used to determine a single point between two extremes, ...''*

Also his construction was very clear. We can see this figure below (p.206/13).

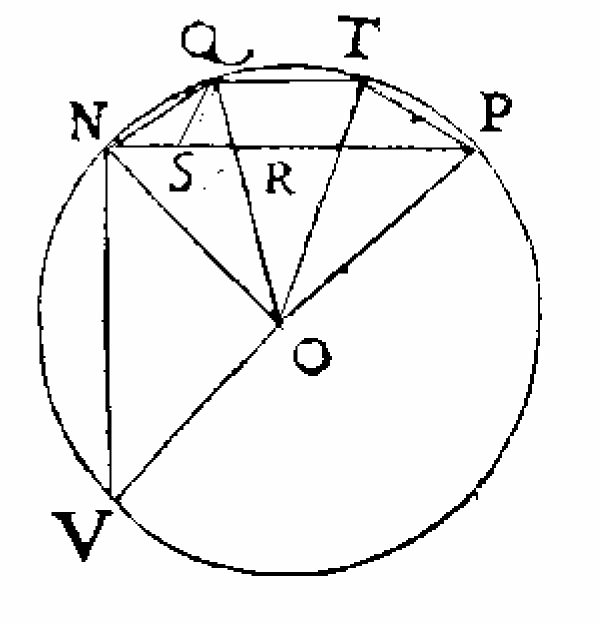


Figure 19: Descartes' trisection method

For example the equation $x^3 = px + q$, with $p, q > 0$ and

Assume that the condition:

ie) $\frac{1}{4}q^2 < \frac{1}{27}p^3$

The chord NP is equal to $\frac{3q}{p}$ in the circle $NQPV$. We can observe that the chord NP is less than the diameter of circle $NQPV$. ie) $\frac{3q}{p} < 2\sqrt{\frac{p}{3}}$ and divide each of the two arcs NQP and NVP into three equal parts. Also Descartes stated that NQ and NV are two true roots of the equation. The positive roots of the trisection are gk and GK (g and G being on the opposite of the axis from E) and the one negative root is FL. Moreover, he stated in the smaller root gk is equal to the NQ on the trisecting arc NP and the other root GK is equal to the NV on the corresponding to trisecting arc NVP, and the negative root FL is equal to $NQ + NV$. Although Descartes did not explicitly mention it here, his other case makes clear. But he did not give a proof. For example, we can observe that the equation $x^3 = 3x - q$ may be obtained from the equation $x^3 = 3x + q$ by transforming the latter into an equation whose roots have the opposite signs. That is, the roots of equation $x^3 = 3x - q$ are the false roots of equation $x^3 = 3x + q$ and vice-versa. Therefore $FL = NQ + NP$ is the true root.

The editors of Descartes' geometry quote Rabuel's proof in a footnote page 208. I explain Rabuel's proof here.

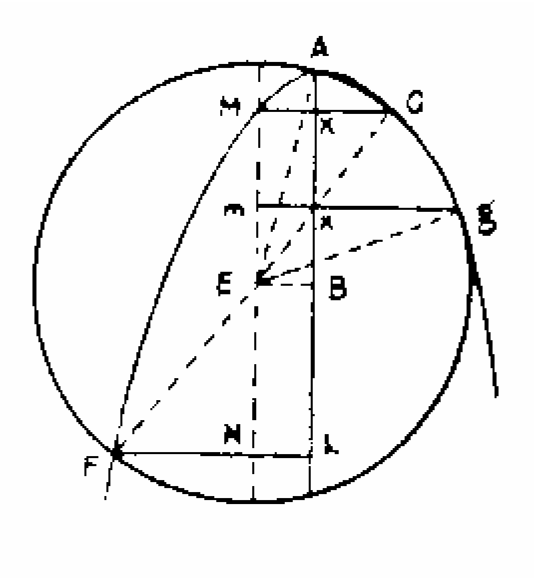


Figure 20: Rabuel's construction of third degree roots

We can assume $AB = p$, $EB = q$, $GK = y$, $gk = z$, and $FL = x$

Then $GM = y + q$, $gm = z + q$, and $FN = x - q$

I would like to consider the latus rectum 'a' of this given parabola.

So, $GK = y$ this implies $AK = \frac{y^2}{a}$

$gk = z$ this implies $Ak = \frac{z^2}{a}$

$FL = x$ this implies $AL = \frac{x^2}{a}$

And simply calculate $ME = AB - AK = p - \frac{y^2}{a}$

Similarly, $mE = p - \frac{z^2}{a}$

$NE = \frac{x^2}{a} - p$

And also, $AE^2 = GE^2 = Eg^2 = EF^2$ (The square of the radius of given circle)

$$p^2 + q^2 = \left(p - \frac{y^2}{a}\right)^2 + (y+q)^2 = \left(p - \frac{z^2}{a}\right)^2 + (z+q)^2 = \left(\frac{x^2}{a} - p\right)^2 + (x-q)^2$$

We take, $p^2+q^2 = p^2-2p\dfrac{y^2}{a}+\dfrac{y^4}{a^2}+y^2+2yq+q^2$

$$2p\frac{y^2}{a}=\frac{y^4}{a^2}+y^2+2yq$$

$$2apy = y^3+a^2y+2a^2q \text{ ……………… (1)}$$

And similarly, $2apz = z^3+a^2z+2a^2q$ ……….(2) and $2apx = x^3+a^2x-2a^2q$ …….(3)

(1) - (2) $\Rightarrow$ $2ap(y-z)=y^3-z^3+a^2(y-z)$

$\Rightarrow$ $2ap=(y^2+yz+z^2)+a^2$ ………… (4)

Similarly, $2ap=(y^2-yx+x^2)+a^2$ ……………(5)

(4) – (5) $\Rightarrow$ $0=(y^2+yz+z^2)-(y^2-yx+x^2)$

$\Rightarrow (y^2+yz+z^2)=(y^2-yx+x^2)$

$\Rightarrow (yz+z^2)=(-yx+x^2)$

$\Rightarrow yz+yx=x^2-z^2$

$\Rightarrow$ $y(z+x)=(x+z)(x-z)$

$\Rightarrow (x+z)[y-x+z]=0$

$\therefore$ $x=y+z$ or $x=-z$

That is, $FL=GK+gk$ or $FL=-gk$

Rabuel did not comment on the second case ($FL=-gk$), but in this case parabola axis AL will be fall into the diameter MN of the circle. That is $EB=q=0$.

Further, I explain the third degree equation in modern terms here.

Assume that $GK=x$, and $AK=y$ then $y=x^2$ because G is on the parabola (see figure 18).

We take the parabola $y=x^2$ with vertical axis and latus rectum is equal to 1.

But G is also on the circle and the centre E of the coordinates are $\left\{\frac{1}{2}q,\frac{1}{2}(p+1)\right\}$.

The equation of the circle is $\left[x-\frac{1}{2}q\right]^2+\left[y-\frac{1}{2}(p+1)\right]^2=d^2$

$$x^2+y^2-qx-(p+1)y+\left[\frac{1}{4}q^2+\frac{1}{4}(p+1)^2\right]=d^2$$

$$x^2 + y^2 - qx - (p+1)y = 0, \text{ where } \left[\frac{1}{4}q^2 + \frac{1}{4}(p+1)^2\right] = d^2$$

We take the parabola $y = x^2$ and ………………………… (1)

Then take the circle $x^2 + y^2 - (1+p)y + qx = 0$ …………... (2)

Substitute (1) by (2) this implies

$$x^2 + x^4 - (1+p)x^2 + qx = 0$$

$$x^4 - px^2 + qx = 0$$

$$x(x^3 - px + q) = 0$$

$$x = 0 \text{ or } x^3 - px + q = 0$$

$$x^3 = px - q$$

The solution from the simple calculation is $x^3 = px - q$.

Descartes gave only for one of his case distinctions (namely, $+p$ and $-q$) and left the other cases to the reader. This example of third degree equation is easy to understand for a modern student today.

4.2.2 Numerical example:

I would like to give a numerical example here and I shall construct the third degree equation by intersection of a circle and a parabola. Also I shall use the mathematica program and find the roots too. The standard form of the third degree equation is $x^3 = \pm px \pm q$ ($a = 1$).

I consider the third degree equation: $x^3 = 7x - 2$

Where $p = 7, q = 2,$ and I consider the value $a = 1$. so, the equation of the parabola is $x^2 = y$.

Now I calculate the values:

$$AD = \frac{1}{2}(p+a) = \frac{1}{2}7 + \frac{1}{2} = 4 \text{ and } DE = \frac{1}{2}q = \frac{1}{2}2 = 1$$

Then I consider the right triangle is ΔEDA.

$$AE^2 = AD^2 + DE^2 = 16 + 1$$

$$AE = \sqrt{17}$$

That is, the radius of the circle is AE. We assume the point A is an origin for the (x,y)-coordinate system along the horizontal direction GK and the vertical axis of parabola, respectively.

Let $GK = x$, $AK = y$ and the point E is (-1, 4).

Then the equation of the circle is $(x+1)^2 + (y-4)^2 = 17$

$$\text{ie. } x^2 + y^2 + 2x - 8y = 0$$

Now I want to solve this equation. So, I choose these two curves, circle: $x^2 + y^2 + 2x - 8y = 0$ and parabola: $x^2 = y$

I consider the circle and parabola:

Circle: $x^2 + y^2 + 2x - 8y = 0$ …………….(a) and

Parabola: $x^2 = y$ …… (b)

First, I would like to plot the two graphs. So, I use the mathematica program commands given below.

g1=Graphics[Circle[{-1, 4}, $\sqrt{17}$];

g2=Plot[x^2, { x , -10, 10}, DisplayFunction→Identity];

Show[g1,g2, AspectRatio→Automatic, PlotRange→{-15,15},

Axes→True, DisplayFunction→$DisplayFunction];

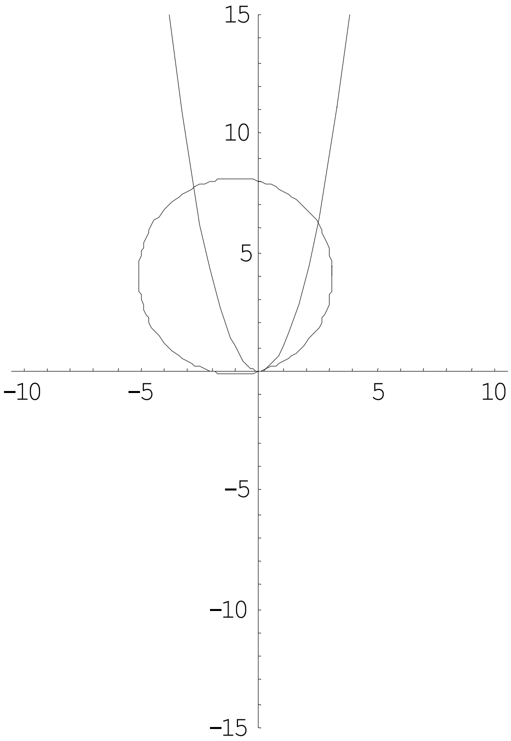


Figure 21: Numerical example for the construction of third degree equations (1)

The circle $x^2 + y^2 + 2x - 8y = 0$ intersects the parabola $x^2 = y$ at four real points. Now we solve the intersection points. We obtain a numerical approximation given below. I use the mathematica program command also given below.

Nsolve[$y == x^2 \&\& x^2 + y^2 + 2x - 8y == 0$]

{{y→7.71982, x→-2.77846}, {y→6.19656, x→2.48929}, {y→0.0836184, x→0.289169}, {y→0., x→0.}}

We can observe the three intersection points in this figure 21. But mathematica gives a result in four intersection points. This is special case of the fourth degree equation. Because one point {y→0.0836184, x→0.289169} is not visible to our eyes. In this case we could check the mathematica plot on other intervals, if we want to have further information.

So, we could check this mathematica plot commands:

g1=Graphics[Circle[{-1, 4}, $\sqrt{17}$];

g2=Plot[x^2, { x, -0.5, 0.5}, DisplayFunction→Identity];

Show[g1,g2, AspectRatio→Automatic, PlotRange→{-1,1},

Axes→True, DisplayFunction→$DisplayFunction];

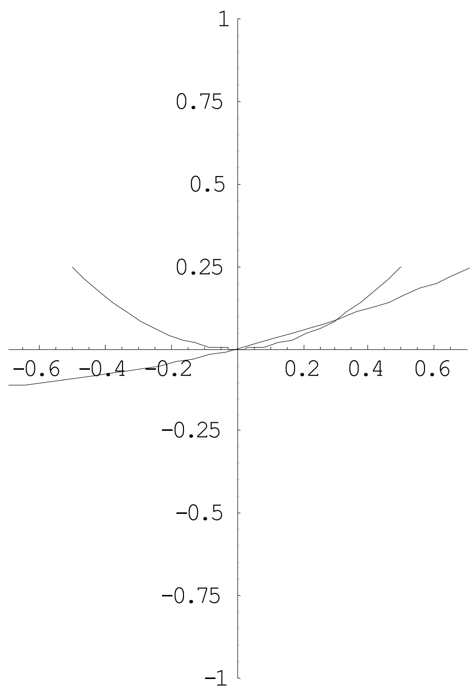


Figure 22: Numerical example for the construction of third degree equations (2)

In this graph is clearly identifying the fourth point.

Now, I shall construct the third degree equation by intersection of a circle ($x^2 + y^2 + 2x - 8y = 0$) and a parabola ($x^2 = y$).

We consider the two equations and substitute (b) in (a), we get the third degree equation

$$x^2 + (x^2)^2 + 2x - 8(x^2) = 0$$

$$x^2 + x^4 + 2x - 8x^2 = 0$$

$$x^4 = 7x^2 - 2x \quad \text{or} \quad x^3 = 7x - 2$$

ie) $x^3 = 7x - 2,$ where $p = 7,$ and $q = 2$

Then I shall find the roots of third degree equation here. So, I use the mathematica command and we get the result below. We can see the three real roots here.

Nsolve [y $= x^3 - 7x + 2 == 0$]

{{x→-2.77846}, {x→2.48929}, {x→0.289169}}

Then I would like to plot this construction of third degree curve $y = x^3 - 7x + 2$ on the interval $-6 \le x \le 6$ given below. We can observe the three real roots and two stationary points in this figure 23 below.

Plot[$x^3 - 7x + 2$, {x,-6,6}];

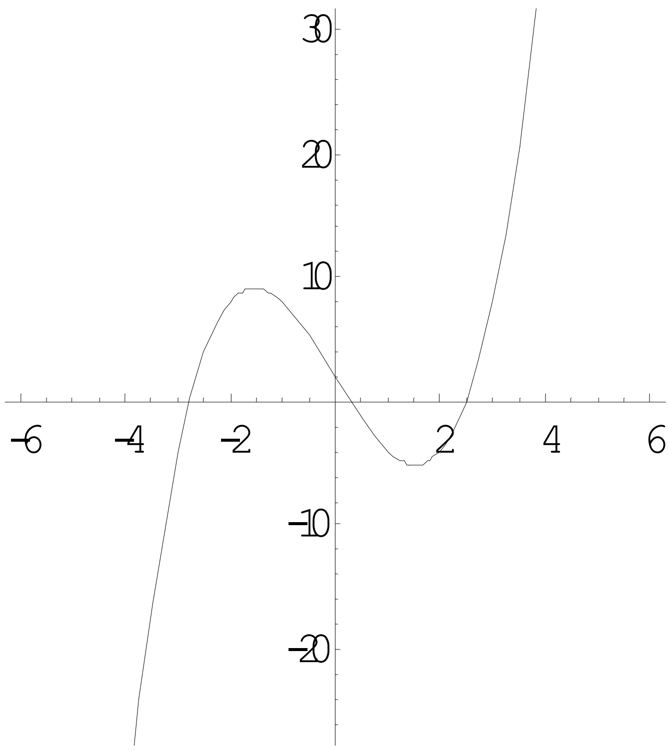


Figure 23: Third degree curve

4.3. The construction of the fifth and sixth degree equations

Descartes reduced the construction of fifth and sixth degree equations to the form:

$$x^6 - px^5 + qx^4 - rx^3 + sx^2 - tx + u = 0 \text{ .... } (*) \qquad [\text{p.}220]$$

Here p, q, r, s, t and u are positive. The coefficients make sure that all real roots are positive; the result also agrees with Descartes sign rule (p.160). For this rule Descartes provided no proof. He may have found it while studying the form of the equations required in his construction of fifth and sixth degree equations (see more below). Descartes also assumes that $q > \frac{1}{4}p^2$. Descartes did not omit the second term of this equation because the equation depends on his assumption. He did not comment on why he chose this equation. Furthermore, he avoided the complicated + / - case distinctions for sixth degree equation [(*)]. See the figure below (p.222/34).

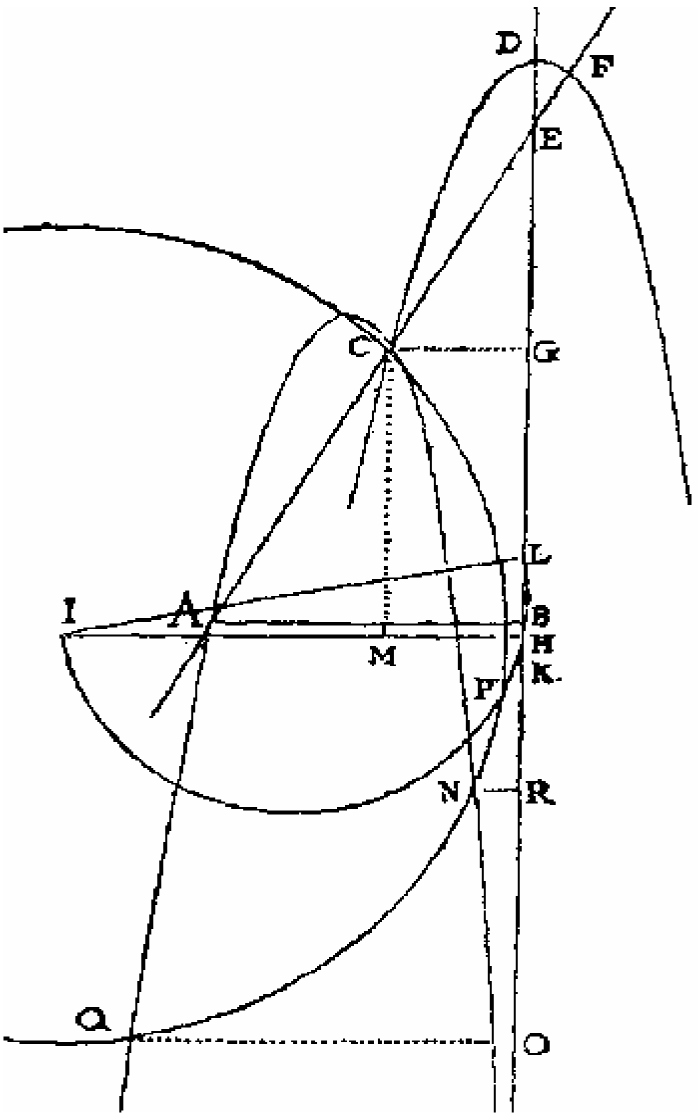


Figure 24: The construction of sixth degree equations

4.3.1 How the solutions of (*) are constructed:

First a summary of the construction (see also the picture), which will afterwards be explained in detail:

By the equation (*) a (traditional) parabola with vertex point D and a fixed point E on its principal axis are determined which are assumed to lie in a plane and move (together) downwards with it such that its principal axis is always overlapping with a given and fixed straight line BK in another plane underneath it, where the distance between B and K does not matter. In this lower plane (which can be imagined as a sheet of paper lying below the first one), in addition to the fixed straight line BK, two points A and I are determined by the coefficients of the equation and by the point B. The connecting straight line between A and E (which can be imagined as a ruler rotating around the fixed point A) intersects with the moving parabola at varying points C and thus defines a curve $QACN$ in the lower plane which we call "Cartesian parabola" and which can be shown to be a third degree algebraic curve.

Descartes drew only one branch of the curve (Cartesian parabola); the other is not involved in his construction.

Around point I in the lower plane a circle can be drawn whose radius is also determined by the coefficients of the equation (*). Now it can be shown (Descartes does not prove it in the book and leaves it to the reader, but he was doubtless able to do it at least for special cases, as we will do it below) that the intersection between the circle and the "Cartesian parabola" gives the roots of the equation (*), which are the rectangular distances of these intersection points to the given straight line BK, which overlaps during the entire construction with the principal axis of the original (traditional) parabola.

The construction requires certain auxiliary quantities, in particular a half circle ILP in the middle of the figure.

Now I describe the construction in detail which one finds on pages 223/24 in *La Géométrie*. Note that the choice of the parameters for the parabola and the circle in dependence on the equation (*) seems at first sight rather unmotivated and is justified only afterwards by showing that the resulting points of intersection deliver the solutions. I do not know why Descartes did not say how he found the values of the parameters n, DE, LH, and LP.

The principal parameter of the traditional parabola, we call n, which is determined by the given equation (*) in the following way:

$$n = \sqrt{\frac{t}{\sqrt{u}} + q - \frac{1}{4}p^2}$$

Now Descartes constructs a point A in the lower plane with the rectangular distance $\frac{p}{2}$ from the straight line BK:

$$AB = \frac{p}{2}$$

Descartes then defines the point E on the principal axis of the parabola by

$$DE = \frac{2\sqrt{u}}{pn}$$

Because A and E are now defined there is also the straight line AE defined for any position of the moving point E.

Descartes is therefore now able to "construct" the "Cartesian parabola", by assuming it to be the collection of all points of intersection between the (rotating) straight lines AE and the moving (traditional) parabola. These points of intersection result in a curve in the lower plane which later was called the "Cartesian parabola", and which is of course in its concrete form (parameters) depending on the equation (*).

Please note that the Cartesian parabola is not "constructed" in the traditional sense with ruler and compass (step by step connecting points and drawing circles around points which have been constructed before) but is the ideal collection of infinitely many constructions, performed with varying points (E) which can be arbitrarily chosen on a straight line BK (here comes the mechanical notion of "movement" in). This is a much looser notion of construction of a geometrical figure or a curve than the traditional one with ruler and compass. One has also to remark that already the assumption of the existence of the traditional parabola CDF is the result of a similar generalized (pointwise) construction. Thus the Cartesian parabola results from an iterated generalized construction which finds its expression in the fact that it is a third degree algebraic curve, while the traditional parabola is a second degree algebraic curve.

Now the Cartesian parabola being "constructed" Descartes performs some more traditional constructions of half-circles and circles because he wants to find a curve (circle) to intersect with the Cartesian parabola which delivers the roots of the original equation (*).

Descartes defines a point L in the lower plane, which lies above the fixed point B in a distance equal to the length DE in the upper plane, that means

$$BL = DE = \frac{2\sqrt{u}}{pn}$$

From L he goes in the opposite direction downwards and constructs a point H on the fixed straight line in the lower plane with the distance

$$LH = \frac{t}{2n\sqrt{u}}$$

Then Descartes erects in H the perpendicular to the left and constructs the point I in the lower plane at the distance

$$HI = \frac{m}{n^2} \text{ where } m = \frac{r}{2} + \sqrt{u} + \frac{pt}{4\sqrt{u}}$$

Descartes then joins the fixed points I and L in the lower plane, divides the connecting straight line in the middle and constructs the half circle on IL (see figure 24). Descartes constructs a point P on the periphery of the half circle just constructed in a distance LP from point L with the following value:

$$LP = \sqrt{\frac{s + p\sqrt{u}}{n^2}}$$

Finally, Descartes takes the distance between the constructed points I and P as the radius of another circle which he draws through P. The circle intersects with the Cartesian parabola in several points, on the figure called Q, N, C and another one above C which does not have a name in Descartes' picture.

Descartes now says page 224/405 without proof that the perpendicular distances $CG; NR,$ and QO of these points of intersection from the straight line BK are the roots of the equation (*) which he is looking for.

'' This circle will cut or touch the curve ACN in as many points as the equation has roots; and hence the perpendiculars $CG; NR, QO$, and so on, dropped from these points upon BK, will be the required roots. This rule never fails nor does it admit of any exceptions.''

In the remaining part of this paragraph I want to show, what is missing in Descartes, namely that the values of these geometric lengths indeed satisfy the sixth degree equation we were starting from.

I go partly back to Descartes (p.84) insofar as I denote the perpendicular distances by the letter x for the unknown.

Furthermore, I shall explain Descartes' proof of the sixth degree equation here.

We take $GC = x$

Since, $n : GC = GC : GD$

Then $GD = \frac{x^2}{n}$

But $DE = \frac{2\sqrt{u}}{pn}$ and $GE = GD - DE = \frac{x^2}{n} - \frac{2\sqrt{u}}{pn}$

Since, $AB : BE = GC : GE$ and $AB = \frac{1}{2}p$

This implies, $BE = \frac{px}{2n} - \frac{\sqrt{u}}{nx}$

Then we know $BL = DE$

So, $BE = DL = \frac{px}{2n} - \frac{\sqrt{u}}{nx}$

Also, $LH = \frac{t}{2n\sqrt{u}}$ and $DL = \frac{px}{2n} - \frac{\sqrt{u}}{nx}$

Therefore, our aim of the calculation is the line segment GH.

So, $GH = DH - GD$

$$= (LH + DL) - GD \quad [\therefore DH = LH + DL]$$

$$GH = \left[\frac{t}{2n\sqrt{u}} + \frac{px}{2n} - \frac{\sqrt{u}}{nx}\right] - \frac{x^2}{n}$$

$$GH = \frac{-x^3 + \frac{1}{2}px^2 + \frac{tx}{2\sqrt{u}} - \sqrt{u}}{nx}$$

Then the square of GH and we get,

$$GH^2 = \frac{x^6 - px^5 + \left(\frac{1}{4}p^2 - \frac{t}{\sqrt{u}}\right)x^4 + \left(2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - p\sqrt{u}\right)x^2 - tx + u}{n^2x^2}$$

Also, again $HI = \frac{r}{2n^2} + \frac{\sqrt{u}}{n^2} + \frac{pt}{4n^2\sqrt{u}}$ and for brevity

$= \frac{m}{n^2}$ where $m = \frac{r}{2} + \sqrt{u} + \frac{pt}{4\sqrt{u}}$

And the right triangle ΔLIH ,

$$IL^2 = IH^2 + HL^2$$

$$= \frac{m^2}{n^4} + \frac{t^2}{4n^2u}$$

And also the right triangle ΔLIP ,

$$IP^2 = IL^2 - LP^2$$

$$= \frac{m^2}{n^4} + \frac{t^2}{4n^2u} - \frac{s}{n^2} - \frac{p\sqrt{u}}{n^2}$$

$IP = R$ is the radius of the circle CPN by construction.

Draw CM , where a right angle is $\angle CMI$

Since, $MI = IH - MH$

$$= \frac{m}{n^2} - x$$

It results $MI^2 = \frac{m^2}{n^4} - \frac{2mx}{n^2} + x^2$

Therefore the right triangle ΔICM ,

$$CM^2 = IC^2 - MI^2$$

$$= \frac{t^2}{4n^2u} - \frac{s}{n^2} - \frac{p\sqrt{u}}{n^2} + \frac{2mx}{n^2} - x^2$$

$$CM^2 = \frac{-n^2x^4 + 2mx^3 - p\sqrt{u}x^2 - sx^2 + \frac{t^2}{4u}x^2}{n^2x^2}$$

But $n^2 = \frac{t}{\sqrt{u}} + q - \frac{1}{4}p^2$, multiplying by x^4 then we get

$$n^2x^4 = \frac{t}{\sqrt{u}}x^4 + qx^4 - \frac{1}{4}p^2x^4$$

Also, $m = \frac{r}{2} + \sqrt{u} + \frac{pt}{4\sqrt{u}}$, multiplying by $2x^3$ then we get

$$2mx^3 = rx^3 + 2\sqrt{u}x^3 + \frac{pt}{2\sqrt{u}}x^3$$

Then we get,

$$CM^2 = \frac{\left(\frac{1}{4}p^2 - q - \frac{t}{\sqrt{u}}\right)x^4 + \left(r + 2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 - p\sqrt{u}x^2 - sx^2 + \frac{t^2}{4u}x^2}{n^2x^2}$$

$$CM^2 = \frac{\left(\frac{1}{4}p^2 - q - \frac{t}{\sqrt{u}}\right)x^4 + \left(r + 2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - s - p\sqrt{u}\right)x^2}{n^2x^2}$$

But the square of GH is equal to square of CM.

So,

$$\frac{x^6 - px^5 + \left(\frac{1}{4}p^2 - \frac{t}{\sqrt{u}}\right)x^4 + \left(2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - p\sqrt{u}\right)x^2 - tx + u}{n^2x^2} =$$

$$\frac{\left(\frac{1}{4}p^2 - q - \frac{t}{\sqrt{u}}\right)x^4 + \left(r + 2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - s - p\sqrt{u}\right)x^2}{n^2x^2},$$

$$\Rightarrow x^6 - px^5 + \left(\frac{1}{4}p^2 - \frac{t}{\sqrt{u}}\right)x^4 + \left(2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - p\sqrt{u}\right)x^2 - tx + u =$$

$$\left(\frac{1}{4}p^2 - q - \frac{t}{\sqrt{u}}\right)x^4 + \left(r + 2\sqrt{u} + \frac{pt}{2\sqrt{u}}\right)x^3 + \left(\frac{t^2}{4u} - s - p\sqrt{u}\right)x^2,$$

or

$$x^6 - px^5 + qx^4 - rx^3 + sx^2 - tx + u = 0$$

The line segments $CG, NR, QO, ...$ are the required roots of the equation.

Descartes explained that in his way he found all roots. All roots of a sixth degree equation were imaginary, if the circle (CPN) of radius R was very small and does not intersect the Cartesian parabola $QACN$ at any point.
Descartes says (p.227):

> *''The circle IP will in general cut the curve ACN in six different points, so that the equation can have six distinct roots. But if it cuts it in fewer points, this indicates that some of the roots are equal or else imaginary''*

If all the roots were positive, only one branch of Cartesian parabola would be really used in the construction, namely, the branch with the local extreme. Descartes already explained that in the case of alternating coefficients in the equation the roots are all positive. Also look in this respect shortly on "Descartes rule of sign", which is given without proof in *La Géométrie* on page 160:

> *''We can determine also the number of true and false roots that any equation can have, as follows: An equation can have as many true roots as it contains changes of sign, from + to - or from - to +; and as many false roots as the number of times two + signs or two - signs are found in succession.'' (Géométrie, p.160)*

This is the well known "Descartes rule of sign". However, it was known before his time by Thomas Harriot. Harriot gave it in his ''Artis analyticae praxis'' in London on 1631. Also the historian M. Cantor said that Descartes may have learned it from Cardan's writings. But Descartes has stated it first it as a general rule (see Cantor, Vol.II (1) p. 496 and 725)
The circle intersects the Cartesian parabola in at maximum of four points; the other two roots were equal or imaginary. The above statements Descartes did not explicitly argue for other two roots. Roberval had not accepted the above Descartes statement in 1638. Also, He criticized the Descartes method for construction of the roots of sixth degree equation.

> *''Roberval thought that the circle would intersect the positive branch of Cartesian parabola in at most four points, the other two being provided by the other branch. Descartes denied this and explained for the example in the figure he had chosen a case in which two roots were imaginary because otherwise the intersections of the*

circle and the branch of Cartesian parabola would be so oblique as to make the points of intersection indistinguishable'',

says J.M Bos (2001) p. 418 in his book.

Normally, this sixth degree equation has six roots but not four. Descartes showed that only one branch of Cartesian parabola is being used in his figure. Also, in this case showed that the circle intersects the Cartesian parabola in at maximum four points; the other two roots were equal or imaginary. A third degree curve usually has an inflection point. This is not a maximum point. Descartes had chosen a case in which two roots were imaginary. We can see figure below, curves with inflection points can be imagined to have more than four intersection points with a circle. But Descartes did not clearly explain these cases. Furthermore, see above Descartes' own words (p.227) and Robervals remark. (We can see this figure 19 on the internet: http://www.2dcurves.com/cubic/cubict.html)

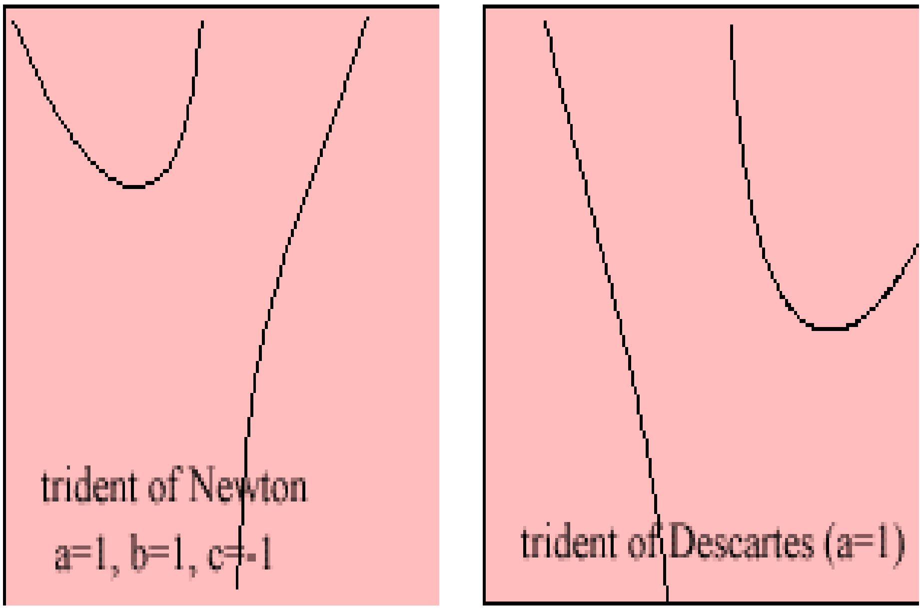


Figure 25: Other branch of the Cartesian parabola

Descartes also discussed an important example of a sixth degree equation, namely, he worked out to the problem of finding four mean proportionals between the line segments a_1 and a_2. But this is a fifth degree equation. We can see below how to find the sixth degree equation by Descartes.

If the line segments a_1, x, y, z, s, a_2 are in continued proportion,

That is, $a_1 : x = x : y = y : z = z : s = s : a_2$, The x, y, z, s will be called the four ''mean proportionals'' between a_1 and a_2.

Algebraically this implies, we get

$$x^2 = a_1 y, y^2 = xz, z^2 = ys \text{ and } xs = a_1 a_2$$

$$\text{ie) } x^5 - a_1^4 a_2 = 0$$

Then the line segment x of fourth mean proportional between two given line segments a_1 and a_2 are known. Descartes used in this method to solve the one example of sixth degree equation. We can see below.

I describe Descartes' idea here.

The relate to an equation was $x^5 - a_1^4 a_2 = 0$

Descartes changed into the required form of sixth degree equation by multiplying with x and substituting $x = y - a_1$.

This implies, $x(x^5 - a_1^4 a_2) = 0$

$$x^6 - a_1^4 a_2 x = 0$$

$$(y-a)^6 - a_1^4 a_2 (y-a) = 0$$

And we get the sixth degree equation,

That is, $y^6 - 6a_1 y^5 + 15a_1^2 y^4 - 20a_1^3 y^3 + 15a_1^4 y^2 - (6a_1^5 + a_1^4 a_2) y + (a_1^6 + a_1^5 a_2) = 0.$

Therefore, we have to take $p = 6a_1$ and $q = 15a_1^2$

That is, $q - \frac{1}{4} p^2 > 0$ (obvious)

And we get, $AB = \frac{1}{2} p = 3a_1$ and the principal parameter n is

$$n = \sqrt{\frac{t}{\sqrt{u}} + q - \frac{1}{4} p^2}$$

$$n = \sqrt{\frac{6a_1^3 + a_1^2 a_2}{\sqrt{a_1^2 + a_1 a_2}} + 6a_1^2}$$

And we get, $DE = \frac{2\sqrt{u}}{pn} = \frac{2a_1\sqrt{a_1^2 + a_1 a_2}}{3n}$. Then the curve $QACN$,

We have, $LH = \frac{t}{2n\sqrt{u}} = \frac{6a_1^3 + a_1^2 a_2}{2n\sqrt{a_1^2 + a_1 a_2}}$ and so on.

We can draw the perpendiculars CG and NR, and therefore the two roots of the above equation are CG and NR. The small length, NR we have to take a_1, and CG must be y. Then $x = y - a_1 = CG - NR$, the first of the required mean proportional.

4.3.2 Numerical example:

I would like to give a numerical example for a sixth degree equation here. I shall find this construction and solution by the mathematica program. Also, I find the roots of sixth degree equation could be constructed by the intersection of a circle and a Cartesian parabola. I shall give one example here. I think this example is easy to understand for people today.

The standard form of the sixth degree equation is $x^6 - px^5 + qx^4 - rx^3 + sx^2 - tx + u = 0$.

I consider the sixth degree equation: $x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1 = 0$ ... (a)

Where $p = 2, q = 3, r = 4, s = 1, t = 2$ and $u = 1$ and $4q > p^2$.

The principal parameter of the traditional parabola, we call n, which is determined by the given equation (a) in the following way:

$$n = \sqrt{\frac{t}{\sqrt{u}} + q - \frac{1}{4}p^2}$$

$$= \sqrt{\frac{2}{\sqrt{1}} + 3 - \frac{1}{4}2^2}$$

$$=2$$

So, the equation of the traditional parabola with the shifting point (0,0) assumed in D: This equation should be $x^2 = 2y$.

Now Descartes constructs a point A in the lower plane with the rectangular distance $\frac{p}{2}$ from the straight line BK :

$$AB = \frac{p}{2} = 1$$

Descartes then defines the point E on the principal axis of the parabola by

$$DE = \frac{2\sqrt{u}}{pn} = \frac{2\times\sqrt{1}}{2\times 2} = \frac{1}{2}$$

Because A and E are now defined there is also the straight line AE defined for any position of the moving point E.

Let $CG = x$, $GH = CM = y$ and I choose H as the fixed point $(0,0)$ of the coordinate system.

First I find the equation of the circle. So, I consider the value of IP.

So, $IP = \sqrt{\frac{m^2}{n^4} + \frac{t^2}{4n^2u} - \frac{s}{n^2} - \frac{p\sqrt{u}}{n^2}}$ and

$$HI = \frac{m}{n^2} \text{ where } m = \frac{r}{2} + \sqrt{u} + \frac{pt}{4\sqrt{u}}$$

$$= \frac{4}{2} + \sqrt{1} + \frac{2\times 2}{4\times\sqrt{1}}$$

$$=4$$

$$IP = \sqrt{\frac{16}{16} + \frac{4}{4\times 4\times 1} - \frac{1}{4} - \frac{2\times 1}{4}} = \sqrt{\frac{1}{2}}$$

$IP = \sqrt{\frac{1}{2}}$. This is the radius of the circle.

Assume that the point I is $(x_1,0)$.

So, the equation of the circle is $(x - x_1)^2 + (y-0)^2 = \frac{1}{2}$

Now I calculate the value $HI = \frac{m}{n^2} = \frac{4}{4} = 1$

So, $AB = HI = 1$. That is, $x_1 = 1$.

$\therefore$ The point I is $(1,0)$.

Assume that the point B is $(0, y_1)$.

Then $BH = LH - BL = \frac{t}{2n\sqrt{u}} - \frac{2\sqrt{u}}{pn}$

$$\text{ie. } BH = \frac{2}{2\times 2\times 1} - \frac{2\times 1}{2\times 2} = 0$$

So, $y_1 = 0$. That is, $AB = HI$

In general, A and I are different points but in this particular example the points B and H and the points I and A are coinciding, which makes the example simpler.

Then equation of the circle is $(x - x_1)^2 + (y - 0)^2 = \frac{1}{2}$

$$(x-1)^2 + y^2 = \frac{1}{2}$$

So, the equation of the circle is $2x^2 + 2y^2 - 4x + 1 = 0$.

Descartes is therefore now able to "construct" the "Cartesian parabola", by assuming it to be the collection of all points of intersection between the (rotating) straight lines AE and the moving (traditional) parabola. Now I find the equation of the Cartesian parabola.

Also we consider the triangles ΔCAM and ΔCGE are similar. We can see page 24/25 (my previous work).

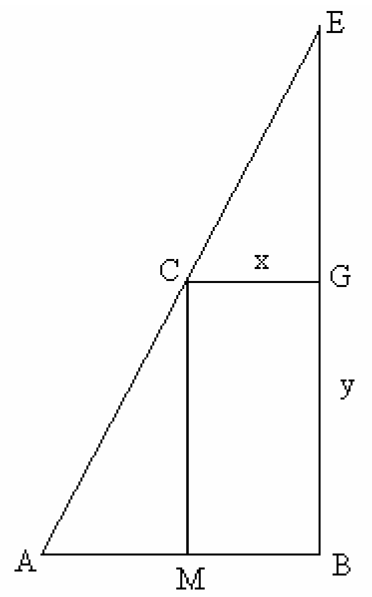


Figure 26: the construction of the third degree curve

So, this implies $\frac{AM}{CM} = \frac{CG}{GE}$

$$\frac{1-x}{y} = \frac{x}{GE} \quad \text{where } AB = 1 \text{ and } GE = GD - DE = \frac{x^2}{n} - \frac{2\sqrt{u}}{pn}$$

$$GE = \frac{x^2}{2} - \frac{1}{2}$$

$$\frac{1-x}{y} = \frac{x}{\frac{x^2}{2} - \frac{1}{2}}$$

$$(1-x)(x^2-1) = 2xy$$

$$-x^3 + x^2 + x - 1 = 2xy$$

This is the Cartesian Parabola.

Now I want to solve this equation. So, I choose these two curves, circle: $2x^2 + 2y^2 - 4x + 1 = 0$ and Cartesian parabola: $2xy = -x^3 + x^2 + x - 1$

I consider the circle and Cartesian parabola:

Circle: $2x^2 + 2y^2 - 4x + 1 = 0$ ……………..(i) and

Cartesian parabola: $2xy = -x^3 + x^2 + x - 1$ …(ii)

First, I would like to plot the two graphs. So, I use the mathematica program commands given below.

g1=Graphics[Circle[{1, 0}, $\sqrt{\frac{1}{2}}$];

g2=Plot[$\frac{-x^3 + x^2 + x - 1}{2x}$, { x , -5, 5}, DisplayFunction $\rightarrow$ Identity];

Show[g1,g2, AspectRatio $\rightarrow$ Automatic, PlotRange $\rightarrow$ {-10,10},

Axes $\rightarrow$ True, DisplayFunction $\rightarrow$ $DisplayFunction];

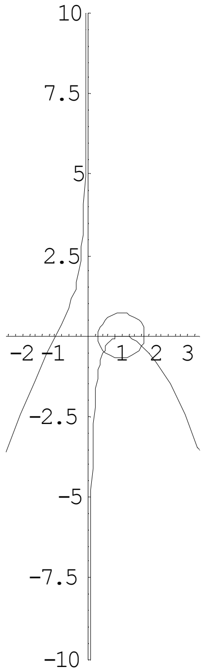


Figure 27: Numerical example for the construction of sixth degree equations

The circle $2x^2 + 2y^2 - 4x + 1 = 0$ intersects the Cartesian parabola $2xy = -x^3 + x^2 + x - 1$ at two real points. Now we solve the intersection points. Because of the complicated structure of the exact solution, we obtain a numerical approximation given below. I use the mathematica program command also given below.

Nsolve[$y == \dfrac{-x^3 + x^2 + x - 1}{2x} \,\&\&\, 2x^2 + 2y^2 - 4x + 1 == 0$]

{{y→-0.320488, x→1.63031}, {y→1.58284 +0.668062 ⅈ, x→0.269125 +1.44681 ⅈ},
{y→1.58284 -0.668062 ⅈ, x→0.269125 -1.44681 ⅈ}, {y→0.806691 -1.1689 ⅈ,
x→-0.315154-0.716979 ⅈ}, {y→0.806691 +1.1689 ⅈ, x→-0.315154+0.716979 ⅈ},
{y→-0.45857, x→0.461749}}

Now, I shall construct the sixth degree equation by intersection of a circle ($2x^2 + 2y^2 - 4x + 1 = 0$) and a Cartesian parabola ($2xy = -x^3 + x^2 + x - 1$).

We consider the two equations and substitute (i) in (ii); we get the sixth degree equation

$$x^2 + \left(\frac{-x^3 + x^2 + x - 1}{2x}\right)^2 - 2x + \frac{1}{2} = 0$$

$$4x^4 + [x^6 - 2x^5 + x^4 - 2x^4 + 4x^3 - 2x^2 + x^2 - 2x + 1] - 8x^3 + 2x^2 = 0$$

ie) $x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1 = 0$

Where $p = 2, q = 3, r = 4, s = 1, t = 2$ and $u = 1$. Also, in this case of sixth degree equation satisfied the Descartes condition for $p^2 < 4q$.

Then I shall find the roots of sixth degree equation here. So, I use mathematica command and we get the result below.

Nsolve[$x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1 == 0$]

{{x→1.63031}, {x→0.269125 +1.44681 ⅈ}, {x→0.269125 -1.44681 ⅈ},

{x→-0.315154+0.716979 ⅈ}, {x→-0.315154-0.716979 ⅈ}, {x→0.461749}}

Also, we can observe the two real roots and others are imaginary. The real roots are 1.63031 and 0.461749. Because, the Cartesian parabola and a circle are intersect of the two real points (see figure 28).

We can see the sixth degree graph given below. Also, we can identify the two real roots 1.63031 and 0.461749 here.

Plot[$x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1$, {$x$,-2,2}];

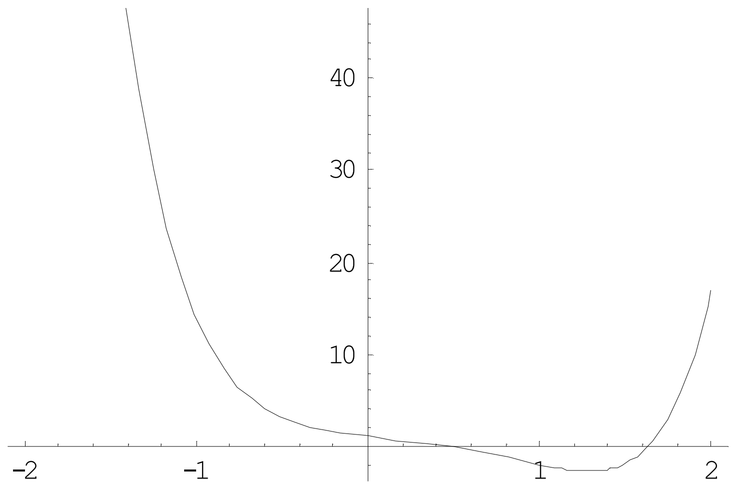


Figure 28: Sixth degree curve

In this figure 28, we can observe one minimum point on this interval $-2 \le x \le 2$. But I shall change the interval, for example $-4 \le x \le 4$, that point is not visible to our eyes on this interval (see appendix, p.83/4).

5. The construction of higher degree equations

Descartes discusses the construction for higher degree equations at the last page of the book III and he says (p.240):

> ''*.... Furthermore, having constructed all plane problems by the cutting of a circle by a straight line, and all solid problems by the cutting of a circle by a parabola; and, finally, all that are but one degree more complex by cutting a circle by a curve but one degree higher than the parabola, it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum; for in the case of a mathematical progression, whenever the first two or three terms are given, it is easy to find the rest....*''

Descartes did not give any further details in his book III and also he did not provide any practical work of this case. It is difficult to understand what Descartes meant by the last line saying '' it is easy to find the rest'' for the generalizing to higher degree equations (p.240).
Also Descartes classified the equations of degree $2n-1$ and $2n$ in his book II (p.56) and he called this classification the n^{th} class of equations (p.48). J.M Bos (p.372) stated that Descartes probably envisaged that equations of degrees $2n-1$ and $2n$ should be constructed by the intersection of a circle and a curve of degree n. That is, all $2n$ ($2 \times n$) equations can be solved by intersection of a circle (degree 2) and a curve of degree n. Further, in 1779, Etienne Bezout proved that the two curves of degree m and n, respectively, intersect in general in $m \times n$ points (J.M.Bos, p.360). That means Descartes' assumption was partly confirmed by a proof of Etienne Bezout (1730-1783).

Table 1: Descartes construction of higher degree equations by using lower degree curves

Degree of the equations	Name for the problems	Curve needed to construct the equations	Class of curve
1	plane	line	unclear
2	plane	circle and line	unclear
3	solid	circle and parabola	first
4	solid	circle and parabola	first
5	Super solid	circle and Cartesian parabola	second
6	Super solid	Circle and Cartesian parabola	second
7, 8, 9,……	-	higher	3, 4, 5,…..so on

6. Conclusion

Descartes' Geometry consists of three books. Book I is about geometrical interpretation of the operations of arithmetic ($+,-,\times,\div$, and square root extraction), construction of plane problems, and the general analysis of Pappus' problem (deriving from a Pappus equation). Book II is about a full solution of Pappus' problem in the three and four lines, and a explanation of two cases of the problem in five lines, the explanation of the curves which are acceptable or not acceptable in geometry, the acceptability of pointwise construction of curves, the study of ovals, and curves on non-plane surfaces.

Book III deals with simplicity of problems, solutions, and curves, and gives Descartes' standard non-plane constructions. In the second part of book III he gives the standard constructions for equations of third and fourth degree and for those of fifth and sixth degree. In the last page of book III he stated that how to extend the general rule of construction to equations of ever higher degree (J.M.Bos (2001), p.290/91).

Descartes did not find the clear expression for his geometry until the *Géométrie* of 1637. According to this vision geometry can and should be structured, and the confusing jumble of problems, methods and solutions, in which it is impossible to know where the problems end and the solutions start, can and should be cleared up.

Descartes first studied Pappus' problem in the late part of 1631 and early part of 1632. J.M.Bos(2001, p.333) stated that '' Descartes' solutions of Pappus' problem as presented in the *Geometry* was impressive indeed and well suited to convince his readers of the power of his new method and of his own virtuosity in handling it''. The Descartes' solution of Pappus' problem illustrates his strange mixture of clearness and concealment in the geometry. The Descartes' *Géométrie* book was indeed an essay on method; it explained with great clearness a novel method for finding the solution of geometrical problems.

Descartes proved and commented the constructions of four line loci clearly enough, but he did not comment how he determined the location and the parameters of the conic sections from their equation and he only provided the values. He accomplished that his solution of the general three and four line locus problem had an importance beyond the special sphere of the Pappus' problems. He wrote:

''Since all equations of degree not higher than the second are included in the discussion just given, not only is the problem of the ancients relating to three or four lines completely solved, but also the whole problem of what they called the composition of solid loci, and consequently that of plane loci, since they are included under solid loci'' (p. 79).

The tracing procedure of the five line locus by turning a ruler and moving a parabola was exhibited in detail; the curve played an important role in his theory of geometrical construction.
Descartes teaches us about ''construction of equations'' in his geometry book and his contribution to clarifying geometrical constructability was the most influential; almost all mathematicians after him took over his view as described in the *Géométrie*. Descartes' method of geometrical constructions fitted well into his programme of using algebra in geometry. It is here that the construction of equations has it crucial position in the geometrical theory, because it forms the bridge between the application of algebra as a tool in geometry, and the actual geometrical construction (J.M.Bos, 1984, p.338).
Descartes explained how to construct the roots of quadratic equations in his geometry book I. In order to move on to higher degrees he had to explain what he meant by constructional exactness. The general construction for equations of third and fourth degree by means of a circle and a parabola was beautiful and constituted a marked improvement of the then extant methods. Descartes' geometry provided a general construction of the roots of fifth and sixth degree equations and claimed that this construction, together with the one for third and fourth degree equations, showed how the technique could be extended to higher degree equations. Descartes constructed equations of fifth and sixth degrees by circle and Cartesian parabola. He did not explain why he chose the Cartesian parabola with its particular origin from the parabola, and so that choice remained unconvincing because of its arbitrariness. He wrote that '' it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum''. It seems likely that Descartes had a general method in his mind.
J.M.Bos (2001, p.374) formulated the general rule of construction that Descartes presented in his geometry. He wrote:
''Construction in geometry should be performed by the intersection of curves. The curves had to be geometrically acceptable and simplest possible for the problem at hand. Geometrically

acceptable curves were precisely the algebraic ones; their simplicity was to be determined by their degrees. With these premises the procedure for constructing problems was:

1. Confronted with a problem, the geometer should first translate it into its algebraic equivalent, that is, an equation.
2. If the equation involved one unknown only, the problem was a normal construction problem. In order to get the simplest construction, the geometer should reduce the equation to an irreducible one.
3. Then he should rewrite it in a certain standard form appropriate to the standard construction to be used.
4. In the case of equations of degrees six or less, the geometer could use standard constructions explicitly given by Descartes. These constructions then provided the geometrical solution of the original problem.
5. In the case of higher-degree equations, the geometer should work out a higher-order analogue for Descartes' standard constructions. Descartes claimed that it should not be difficult to do so.
6. If the equation arrived at in 1 contained two unknowns, the problem was a locus problem. The geometer could construct points on the locus by choosing an arbitrary value for one of the unknown and dealing with the resulting equation(in which there was only one unknown left) according to items 2-5, thus finding the corresponding value(or values) of the second unknown; the corresponding point (or points) on the locus could then be constructed.''

The power of Descartes' vision has shaped western thought since the seventeenth century, and Descartes was one of the founders of modern thought. But whatever my understanding of Descartes the philosopher may be, his importance for mathematics is clear.

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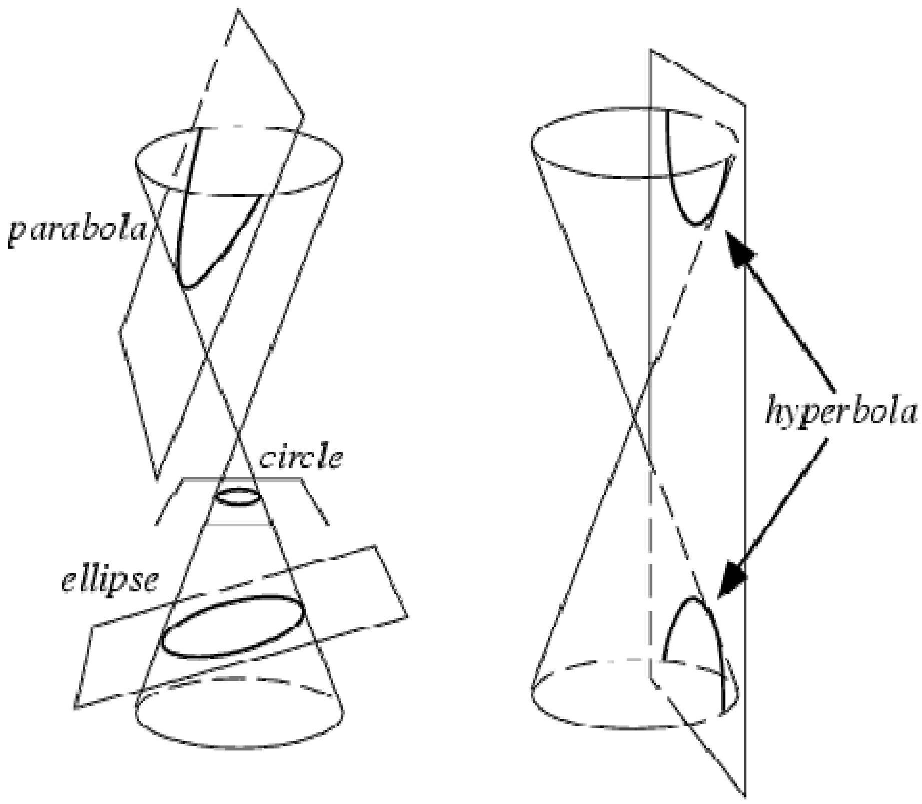
http://www.1911encyclopedia.org/Parabola

http://en.wikipedia.org/wiki/Conic_section

8. Appendix

8.1 Conic section

A **conic section** (or just **conic**) is a curve that can be formed by intersecting a cone (more precisely, a right circular conical surface) with a plane. The conic sections were named and studied as long ago as 200 BC, when Apollonius of Perga undertook a systematic study of their properties.



8.2 Some information on the mathematica program

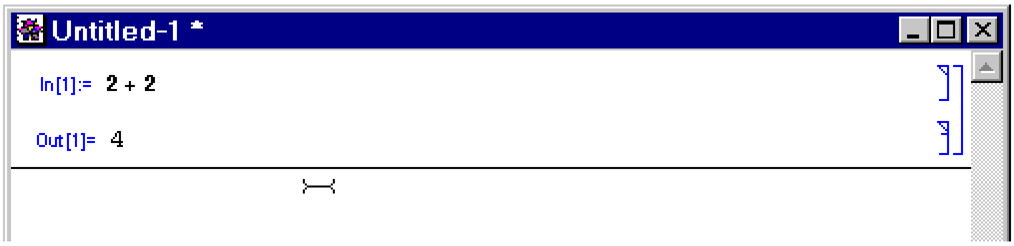
Orientation

When you first start Mathematica, you should see a "splash" screen with the Mathematica logo, version, and license information. When the program loads, you should see several objects on the screen. We will now describe what they are and what they do.

TIP	You may wish to turn on your speakers (or bring headphones if you are in an STC). Mathematica uses audio cues to notify the user of errors, finished calculations, etc.

Notebooks

A notebook is a collection of Mathematica statements, output, and graphics. The concept is like that of a "document" in a word processor. You enter information and commands into the notebook window, and the output (if any) is displayed there.



If the notebook has been modified since it was last saved, an asterisk (*) will appear in the title bar. To save your work, choose File->Save As... or File->Save.

If Mathematica is ready for new input, the cursor will flip sideways (see above). Just start typing to enter information. Try typing this (don't press return yet):

2 + 2

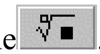
To tell Mathematica to evaluate this expression, hold down Shift and type Return. Since Mathematica is also a word processor, it needs to know if you want to evaluate the expression, or just insert a carriage return-linefeed. This can be quite confusing to the new user.

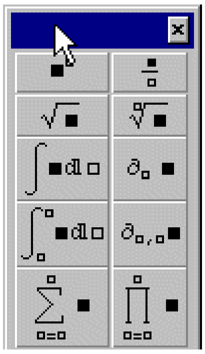
System	Evaluate	Linefeed
Macintosh	Enter *or* Shift-Return	Return
Windows	Shift-Enter	Enter
X	Shift-Return	Return

Next, look at the blue symbols along the right side of the notebook. Each group of statements enclosed by the triangle-brackets ( ) is called a *cell*. The cell is the smallest unit of work in Mathematica. A cell may contain input or output, math or comments, text or graphics.

	*Cells* in Mathematica are like *execution groups* in Maple.

Palettes

You should see a box with a lot of symbols floating on the right side of your Mathematica window. This is called a *palette*. Palettes allow you to easily insert complicated mathematical notation. For example, to compute the cube root of 34, you could click on the . Type **34**, which should appear under the root sign. Then click on the small square above the root, and type **3** in the box. Finally, evaluate the expression. Using the default palette, you can enter fractions, integrals, summations, matrices, subscripts, and most Greek letters. Of course, there are many other palettes available - choose File->Palettes to see a list.



Kernels

Mathematica is actually split into two conceptual pieces, the *front-end* and the *kernel*. When you start Mathematica, you are actually only starting the front-end. The front-end handles input and output to the user, access to the file system, and creates graphics on your screen. Most users will deal primarily with the front-end. The kernel does nearly all computation (excluding graphics rendering). When you evaluate any expression, the kernel does the hard work and sends the results back to the front-end, which then displays it in an attractive format for the user. Most users will run the front-end and the kernel on the same computer. If your computer is connected to a network, you can run the kernel on a more powerful machine, while running the front-end on your favourite computer. To learn how to do this, choose Kernel->Kernel Configuration Options, click Add, and then click Help.

TIP	How to stop a runaway calculation: • Macintosh - Command-Comma or Control-C

	• Windows and X - Alt-Comma or Control-C This will bring up a menu that allows you to view the state of the kernel, abort the calculation, etc.

8.3 Numerical Calculations

Here are some examples.

You could do simple arithmetic with Mathematica. If you wanted to add 1 and 2, you type the input 1 + 2 and hit Return.

In[1]:= 1 + 2

Out[1]= 3

Mathematica can compute exact results, unlike the calculator. The ^ is the notation in Mathematica for rising to a power.

In[2]:= 2^64

Out[2]= 18446744073709551616

You can use the function N to get approximate numerical results. This approximation is given in scientific notation.

In[3]:= 2^64 //N

Out[3]= 1.84467 10^{19}

Some of the common arithmetic operators available in Mathematica are as follows:

^	power
+	add
-	minus or subtract
*	multiply
/	divide

In addition to these Mathematica has a large collection of mathematical functions. Note that all the arguments for these functions have to supply within the square brackets. Also, the functions begin always with a capital letter. Here are some of the
functions:

Sqrt[ ]	square root
Exp[ ]	exponential

Log[ ]	natural logarithm
Sin[ ]	sine function (argument in radians)
ArcSin[ ]	inverse sine function
Abs[ ]	absolute value
Round[ ]	closest integer to the argument
FactorInteger[ ]	prime factors of the argument

8.4 Using Complex Numbers

Mathematica lets you enter complex numbers by including a constant 'I'. This constant I is equal to the square root of -1. Complex number operations can be performed by using the following.

x + I y	complex number x + iy
Re[z]	real part of z
Im[z]	imaginary part of z
Conjugate[z]	complex conjugate of z
Abs[z]	absolute value of z or |z|
Arg[z]	the argument of z

For example,

```
In[10]:= Sqrt[-4]

Out[10]= 2 I

In[11]:= (1 + 2 I)*(1 - 2 I)

Out[11]= 5
```

8.5 Printing graphics

If you want to send graphics to a postscript printer, you can use the PS Print function available in Mathematica.

```
In[19]:= Plot3D[Sin[x y],{x,0,3},{y,0,3}]

Out[19]= -SurfaceGraphics-

In[20]:= PSPrint[%]

Out[20]= -SurfaceGraphics-
```

The print job will be sent to the default printer.

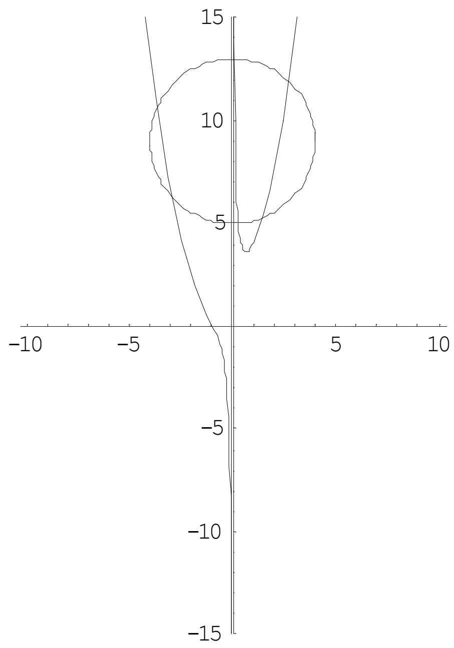
8.6 Other branch of Cartesian parabola

1. g1=Graphics[Circle[{0, 9}, 4];

   g2=Plot[$x^2 + x + 1 + \frac{1}{x}$, { x , -15, 15}, DisplayFunction → Identity];

   Show[g1,g2, AspectRatio → Automatic, PlotRange → {-15,15},

   Axes → True, DisplayFunction → $DisplayFunction];

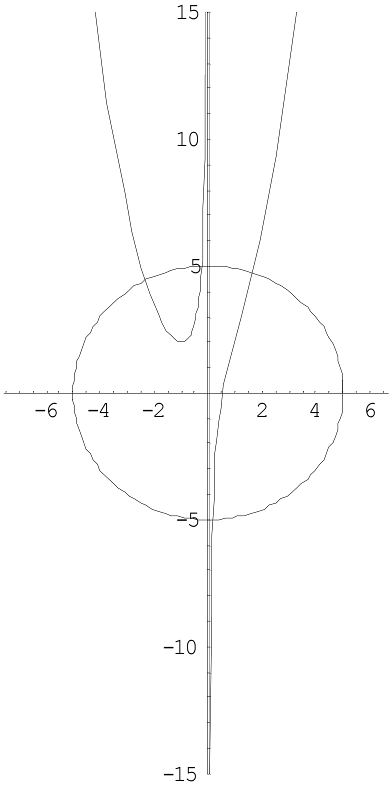


2. g1=Graphics[Circle[{0, 0}, 5];

   g2=Plot[$x^2 + x + 1 - \frac{1}{x}$, { x , -15, 15}, DisplayFunction → Identity];

   Show[g1,g2, AspectRatio → Automatic, PlotRange → {-15,15},

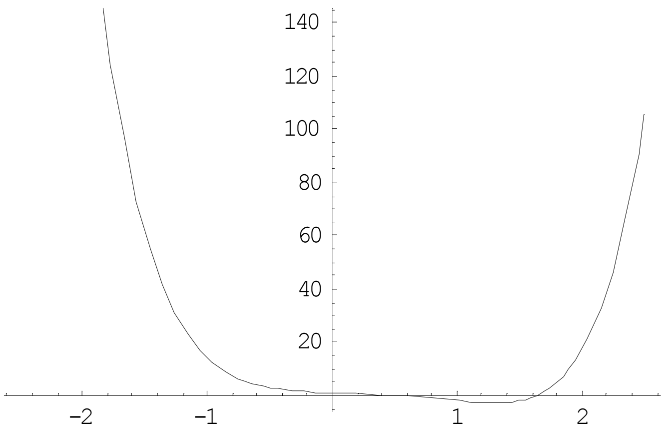
   Axes → True, DisplayFunction → $DisplayFunction];



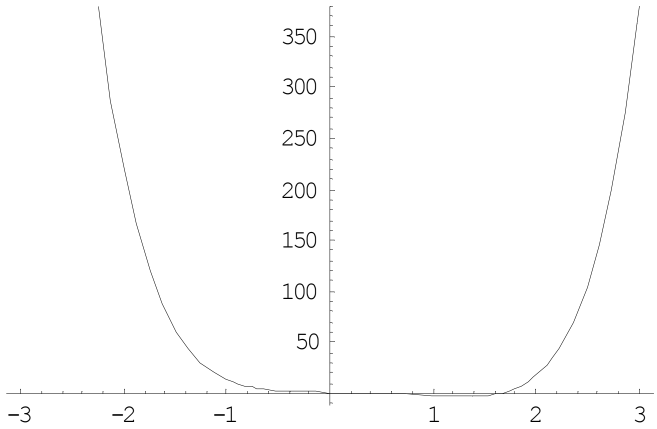
8.7 Sixth degree graph: Numerical example

I plot the sixth degree equation graph on other intervals given below.

1. Plot[$x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1$, { x ,-2.5,2.5}];



2. Plot[$x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1$, {$x$,-3,3}];



3. Plot[$x^6 - 2x^5 + 3x^4 - 4x^3 + x^2 - 2x + 1$, {$x$,-4,4}];