Research Article

Stability and $l_1$-Gain Analysis for Positive 2D Systems with State Delays in the Roesser Model

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1. Introduction

2D systems exist in many practical applications, such as circuits analysis, digital image processing, signal filtering, and thermal power engineering [1–4]. Thus the analysis and synthesis of 2D systems are interesting and challenging problems, and they have received considerable attention; for example, 2D state-space realization theory was researched in [5], the stability and 2D optimal control theory was studied in [6, 7], and $H_{\infty}$ control and filtering problem for 2D systems were addressed in [8–11]. In addition, linear repetitive processes, a distinct class of 2D systems, have also been investigated. For example, the quasi-sliding mode control problem for linear repetitive processes with unknown input disturbance was solved in [12].

The most popular models of two-dimensional (2D) linear systems were introduced by Roesser [13], Fornasini and Marchesini [5, 14], and Kurek [15]. These models have been extended to positive systems in [16–19]. A positive system means that its state and output are nonnegative whenever the initial condition and input are nonnegative [19–21]. Positive 2D systems are needed in many cases such as the wave equation in fluid dynamics and the heat equation which describes the temperature (using thermodynamic temperature scale) in a given region over time and the Poisson’s equation. These facts stimulate the research on 2D positive discrete systems. Reference [22] investigated the choice of the forms of Lyapunov functions for positive 2D Roesser model. The problem of stability analysis for 2D positive systems has been investigated in [17, 23–25]. It should be noted that although positive 2D systems have been discussed in control engineering and mathematics literature recently, there are still many questions which deserve further investigation.

On the other hand, the reaction of real-world systems to exogenous signals is never instantaneous and, always infected by certain time delays. For general systems, even nominal stable systems when were affected by delays may inherit very complex behaviors such as oscillations, instability, and bad performance [26], and delayed systems have attracted many researchers’ attention [27–32]. The reachability, minimum energy control, and realization problem for positive 2D discrete-time systems with delays has been analyzed in [18, 33]. And the stability analysis for 2D positive delayed systems has been investigated in [34–36]. In addition, perturbations and uncertainties widely exist in the practical systems. In some cases, the perturbations and unmodeled errors can be merged into disturbances, which can be supposed to be bounded in the appropriate norms. It is important and necessary to establish a criterion evaluating the disturbance attenuation performance for the positive 2D discrete-time
systems. However, to the best of our knowledge, there has been no literature considering the disturbance attenuation performance for positive 2D systems, which motivates the present study.

In this paper, we will study the problem of delay-dependent stability and $l_1$-gain analysis for positive 2D linear systems with delays. The main theoretical contributions of this paper are as follows (1) We use $l_1$-gain to evaluate the disturbance attenuation performance of positive 2D linear systems. This important performance is firstly considered for positive 2D systems, and a delay-dependent stability criterion of these systems with state delays is developed. (2) Copositive-type Lyapunov function method is firstly used to analyze delay-dependent stability and $l_1$-gain performance for positive 2D linear systems. (3) It is significant to characterize conditions under which the positive 2D delayed system is asymptotically stable. All the developed results are expressed in terms of feasibility testing of LMIs which is computationally tractable.

The paper is organized as follows. In Section 2, problem statement and some definitions concerning the positive 2D linear systems with delays are given. In Section 3, some theorems concerning the delay-dependent stability and $l_1$-gain analysis of positive 2D linear systems are presented. In Section 4, a numerical example is given to illustrate the effectiveness of the proposed results. Finally, concluding remarks are provided in Section 5.

Notations. In this paper, the superscript “$T$” denotes the transpose. The notation $X > Y$ ($X \geq Y$) means that matrix $X-Y$ is positive definite (positive semidefinite, resp.). $A \geq 0$ ($\leq 0$) means that all entries of matrix $A$ are nonnegative (nonpositive). $A > 0$ ($< 0$) means that all entries of matrix $A$ are positive (negative). $R^n$ denotes the set of $n \times m$ real matrices. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $R^n_{\text{pos}}$, $R^n$ denotes the set of vectors with nonnegative entries, and the set of nonnegative integers will be denoted by $Z_n$. The $n \times n$ identity matrix will be denoted by $I_n$. The $l_1$ norm of a 2D signal $w(i,j) = [w_1(i,j), w_2(i,j), \ldots, w_m(i,j)]^T$ is given by

$$
\|w(i,j)\|_1 = \sum_{k=1}^m \|w_k(i,j)\|.
$$

And we say $w(i,j) \in l_1$ if $\|w(i,j)\|_1 < \infty$.

2. Problem Formulation and Preliminaries

Consider the positive 2D Roesser model with state delays [25]:

$$
\begin{bmatrix}
\dot{x}^h(i+1,j) \\
\dot{x}^v(i,j+1)
\end{bmatrix} = A \begin{bmatrix}
x^h(i,j) \\
x^v(i,j)
\end{bmatrix} + A_d \begin{bmatrix}
x^h(i-d_h(i,j), j) \\
x^v(i,j-d_v(j))
\end{bmatrix} + Bu(i,j),
$$

$$
z(i,j) = Hx(i,j) + Lu(i,j),
$$

where $i$ and $j$ are integers in $Z_n$, $x^h(i,j)$ is the horizontal state in $R^n_{\text{pos}}$, $x^v(i,j)$ is the vertical state in $R^n_{\text{pos}}$, $x(i,j)$ is the whole state in $R^n_{\text{pos}}$, $w(i,j) \in R^n_{\text{pos}}$ is the $l_1$ norm bounded disturbance input, $z(i,j) \in R^n_{\text{pos}}$ is the controlled output, and $A, A_d, B, H, L \geq 0$ are system matrices with compatible dimensions. The matrices are

$$
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
A_d = \begin{bmatrix}
A_{d11} & A_{d12} \\
A_{d21} & A_{d22}
\end{bmatrix},
B = \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}.
$$

$d_h(i)$ and $d_v(j)$ are delays along horizontal and vertical directions, respectively. We assume that $d_h(i)$ and $d_v(j)$ satisfy

$$
d_h(i) \leq d_{hL} \leq d_{hT},
$$

$$
d_v(j) \leq d_{vL} \leq d_{vT},
$$

where $d_{hL}, d_{hT}, d_{vL}$ and $d_{vT}$ denote the lower and upper delay bounds along horizontal and vertical directions, respectively. The boundary conditions are defined by

$$
x^h(i,j) = h_{ij}, \forall 0 \leq j \leq z_1, -d_{hT} \leq i \leq 0,
$$

$$
x^h(i,j) = 0, \forall j > z_1, -d_{hT} \leq i \leq 0,
$$

$$
x^v(i,j) = v_{ij}, \forall 0 \leq i \leq z_2, -d_{vT} \leq j \leq 0,
$$

$$
x^v(i,j) = 0, \forall i > z_2, -d_{vT} \leq j \leq 0,
$$

where $z_1 < \infty$ and $z_2 < \infty$ are positive integers, $h_{ij} \in R^n_{\text{pos}}$ and $v_{ij} \in R^n_{\text{pos}}$ are given vectors.

**Definition 1.** The 2D positive system (1a) and (1b) with $w(i,j) = 0$ is said to be asymptotically stable if $\lim_{t \to \infty} X_t = 0$ for all bounded boundary conditions (4), where

$$
X_t = \sup \{\|x(i,j)\| : i + j = l, i, j \geq 1\}.
$$

**Definition 2.** For $\gamma > 0$, the system (1a) and (1b) is said to be asymptotically stable with the $l_1$-gain index $\gamma$, if the following conditions hold.

1. The system (1a) and (1b) with $w(i,j) = 0$ is asymptotically stable.

2. Under zero boundary conditions, that is, $h_{ij} = v_{ij} = 0$ in (4), it holds that

$$
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|z(i,j)\|_1 \leq \gamma \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|w(i,j)\|_1, \forall 0 \neq w(i,j) \in l_1.
$$

**Remark 3.** From (6), we see that $\gamma$ can characterize the disturbance attenuation performance of the system (1a) and (1b). The smaller the $\gamma$ is, the better the disturbance attenuation performance is.
3. Main Results

3.1. Stability Analysis. In this subsection, we focus on the problem of delay-dependent asymptotically stability analysis for the positive 2D discrete linear systems with state delays.

**Theorem 4.** For given positive constants \(d_{id}, d_{hd}, d_{vd}, d_{vdT}\), the positive 2D system \((1a)\) and \((1b)\) with \(w(i, j) = 0\) is asymptotically stable if there exist vectors \(p, q, \zeta_1, \zeta_2, \zeta \in \mathbb{R}^n\), such that

\[
\Phi = \text{diag} \{ \Phi_1, \Phi_2, \ldots, \Phi_n, \Phi'_1, \Phi'_2, \ldots, \Phi'_n, \Phi''_1, \Phi''_2, \ldots, \Phi''_n \} < 0,
\]

where

\[
\begin{align*}
\Phi_k &= \begin{pmatrix}
(a_k^T - E_k) p + (a_k^T + (d_{hd} - d_{id}) E_k)q \\
+ E_k \zeta + (d_{hd}^2 (a_k^T - E_k) - d_{id} E_k) \xi_2 & 1 \leq k \leq n, \\
+ d_{hd}^2 E_k \xi_1, \\
(a_k^T - E_k) p + (a_k^T + (d_{vd} - d_{vdT}) E_k)q \\
+ E_k \zeta + (d_{vdT}^2 (a_k^T - E_k) - d_{vd} E_k) \xi_2 & n+1 \leq k \leq n, \\
+ d_{vd}^2 E_k \xi_1, \\
(a_k^T - E_k) p + (a_k^T + (d_{vd} - d_{vdT}) E_k)q \\
+ E_k \zeta + (d_{vdT}^2 (a_k^T - E_k) - d_{vd} E_k) \xi_2 & 1 \leq k \leq n, \\
+ d_{vd}^2 E_k \xi_1, \\
-E_k \zeta + d_{hd} E_k (\xi_2 - \xi_1), & 1 \leq k \leq n, \\
-E_k \zeta + d_{vd} E_k (\xi_2 - \xi_1), & n+1 \leq k \leq n, \\
-d_{hd} E_k \xi_1, & 1 \leq k \leq n, \\
-d_{vd} E_k \xi_1, & n+1 \leq k \leq n,
\end{pmatrix}
\end{align*}
\]

with \(k \in \mathbb{N} = \{1, 2, \ldots, n\}, E_k = \left[ \begin{array}{c} k-1 \\ 0, \ldots, 0, 1, 0, \ldots, 0, n-k \end{array} \right], \text{ and } a_k(a_k) \text{ represents the } k\text{th column vector of matrix } A(A_d)\).

**Proof.** Choose the following copositive Lyapunov-Krasovskii functional candidate:

\[
V(i, j) = V^h(i, j) + V^v(i, j),
\]

where

\[
\begin{align*}
V^h(i, j) &= \sum_{k=1}^{5} V^h_k(i, j), \\
V^h_1(i, j) &= x^{hT}(i, j) p^h, \\
V^h_2(i, j) &= \sum_{r_i-d_{id}(0)}^{i-1} x^{hT}(r, j) q^h, \\
V^h_3(i, j) &= \sum_{r_i-d_{hd}}^{i-1} x^{hT}(r, j) \zeta^h, \\
V^h_4(i, j) &= \sum_{r_i-d_{vdT}+1}^{i-1} x^{hT}(r, j) q^h, \\
V^h_5(i, j) &= d_{hd} \sum_{r_i-d_{vdT}+1}^{i-1} \eta^{ht}(r, j) \zeta^h, \\
V^v(i, j) &= \sum_{k=1}^{5} V^v_k(i, j), \\
V^v_1(i, j) &= x^{vT}(i, j) p^v, \\
V^v_2(i, j) &= \sum_{s=j-d_{id}(i)}^{1} x^{vT}(i, s) q^v, \\
V^v_3(i, j) &= \sum_{t_j-d_{vd}(j)}^{i-1} x^{vT}(i, t) \zeta^v, \\
V^v_4(i, j) &= \sum_{s=d_{vdT}+1}^{i-1} x^{vT}(i, t) q^v, \\
V^v_5(i, j) &= \sum_{s=d_{vdT}+1}^{i-1} x^{vT}(i, t) \zeta^v, \\
\eta^h(r, j) &= [x^{ht}(r, j) \delta^{ht}(r, j)]^T, \\
\eta^v(i, t) &= [x^{vt}(i, t) \delta^{vt}(i, t)]^T, \\
\delta^h(r, j) &= x^h(r+1, j) - x^h(r, j), \\
\delta^v(i, t) &= x^v(i, t+1) - x^v(i, t),
\end{align*}
\]

with \(p^h, q^h, \zeta^h, \chi^h, \zeta, \zeta_1, \zeta_2, \zeta \in \mathbb{R}^n, p^v, q^v, \zeta^v, \zeta_1, \zeta \in \mathbb{R}^n, \chi^v = [\chi^v_1, \chi^v_2]^T \in \mathbb{R}^{2n}, \text{ and } \chi^v = [\chi^v_1, \chi^v_2]^T \in \mathbb{R}^{2n} \).

Along the trajectory of the system \((1a)\) and \((1b)\), we have

\[
\Delta V(i, j) = V^h(i, j+1) - V^h(i, j) + V^v(i, j+1) - V^v(i, j) = \sum_{k=1}^{5} \Delta V^h_k(i, j) + \sum_{k=1}^{5} \Delta V^v_k(i, j),
\]

where

\[
\begin{align*}
\Delta V^h_k(i, j) &= \sum_{k=1}^{5} \Delta V^h_k(i, j), \\
\Delta V^v_k(i, j) &= \sum_{k=1}^{5} \Delta V^v_k(i, j),
\end{align*}
\]
where

\[
\Delta V_1^h(i, j) = x^{HT}(i + 1, j) q^h - x^{HT}(i, j) p^h,
\]

\[
\Delta V_2^h(i, j) = \sum_{r=i+1-d_H(i+1)}^{i+1} x^{HT}(r, j) q^h - \sum_{r=i-d_H(i)}^{i} x^{HT}(r, j) q^h
\]

\[
= x^{HT}(i + 1, j) q^h - x^{HT}(i - d_H(i), j) q^h + \sum_{r=i+1-d_H(i+1)}^{i} x^{HT}(r, j) q^h
\]

\[
\leq x^{HT}(i + 1, j) q^h - x^{HT}(i - d_H(i), j) q^h + \sum_{r=i+1-d_H(i+1)}^{i} x^{HT}(r, j) q^h.
\]

\[
\Delta V_3^h(i, j) = \sum_{r=i+1-d_H(i)}^{i} x^{HT}(r, j) \zeta^h - \sum_{r=i+1-d_H(i)}^{i} x^{HT}(r, j) \zeta^h
\]

\[
= x^{HT}(i, j) \zeta^h - x^{HT}(i - d_{H'}, j) \zeta^h + \sum_{r=i+1-d_H(i)}^{i} x^{HT}(r, j) \zeta^h.
\]

\[
\Delta V_4^h(i, j) = \sum_{s=-d_H+1}^{i} \sum_{r=i+1+s}^{i} x^{HT}(r, j) q^h - \sum_{r=-d_H+1}^{i} \sum_{r=i+1+s}^{i} x^{HT}(r, j) q^h
\]

\[
= \sum_{s=-d_H+1}^{i} \left[ x^{HT}(i, j) q^h - x^{HT}(i + s, j) q^h \right]
\]

\[
= (d_{H'} - d_H) x^{HT}(i, j) q^h - \sum_{r=i+1-d_H(i+1)}^{i} x^{HT}(r, j) q^h.
\]

\[
\Delta V_5^h(i, j)
\]

\[
= d_{H'} \sum_{s=-d_H}^{i} \sum_{r=i+1+s}^{i} \eta^{HT}(r, j) \zeta^h - d_{H'} \sum_{s=-d_H}^{i} \sum_{r=i+1+s}^{i} \eta^{HT}(r, j) \zeta^h
\]

\[
= d_{H'} \sum_{s=-d_H}^{i} \left( \eta^{HT}(i, j) \zeta^h - \eta^{HT}(i + s, j) \zeta^h \right)
\]

\[
= d_{H'}^2 \left[ x^{HT}(i, j) \ x^{HT}(i + 1, j) - x^{HT}(i, j) \right] \left[ \zeta^h \right]
\]

\[
\Delta V_6^u(i, j) = x^{HT}(i, j) p^u - x^{HT}(i, j) p^u,
\]

\[
\Delta V_7^u(i, j) = \sum_{s=j+1-d_v(s+1)}^{j+1} x^{HT}(i, s) q^u - \sum_{s=j+1-d_v(s)}^{j} x^{HT}(i, s) q^u
\]

\[
= x^{HT}(i, j) q^u - x^{HT}(i, j - d_v(j)) q^u + \sum_{s=j+1-d_v(s+1)}^{j+1} x^{HT}(i, s) q^u
\]

\[
\leq x^{HT}(i, j) q^u - x^{HT}(i, j - d_v(j)) q^u + \sum_{s=j+1-d_v(s+1)}^{j+1} x^{HT}(i, s) q^u.
\]
\[\Delta V^u_s(i, j) = \sum_{s=j+1-d_{uH}}^{j} x^{HT}(i, s) \zeta^u - \sum_{s=j-d_{uH}}^{j-1} x^{HT}(i, s) \zeta^u\]

\[\Delta V^u_s(i, j) = \sum_{s=-d_{uL}}^{-d_{uH}} \sum_{t=j-s}^{j} x^{HT}(i, t) q^v - x^{HT}(i, j) q^v\]

\[\Delta V^u_s(i, j) = \sum_{s=-d_{uH}}^{j} \sum_{t=j-s}^{j-1} x^{HT}(i, t) q^v - x^{HT}(i, j) q^v\]

\[\Delta V^u_s(i, j) = \sum_{s=-d_{uH}}^{j} \sum_{t=j-s}^{j-1} x^{HT}(i, t) q^v - x^{HT}(i, j) q^v\]

Substitute the previously mentioned formulations into (11), and take

\[p = \begin{bmatrix} p^h \\ p^v \end{bmatrix}, \quad q = \begin{bmatrix} q^h \\ q^v \end{bmatrix}, \quad \zeta = \begin{bmatrix} \zeta^h \\ \zeta^v \end{bmatrix}, \quad \zeta_1 = \begin{bmatrix} \zeta_1^h \\ \zeta_1^v \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} \zeta_2^h \\ \zeta_2^v \end{bmatrix}\]

\[D_H = \begin{bmatrix} d_{HH}I_{n_1} & 0 \\ d_{uH}I_{n_2} & 0 \end{bmatrix}, \quad D_L = \begin{bmatrix} d_{HL}I_{n_1} & 0 \\ d_{uL}I_{n_2} & 0 \end{bmatrix}\]

Then we have

\[\Delta V(i, j) = x^{HT}(i, j) \left\{ (A^T - I_n) p + (A^T + D_H - D_L) q + \zeta + D_H^T \left( (A^T - I_n) \zeta_2 + \zeta_1 - D_H \zeta_2 \right) \right\} + x_d^T(i, j) \left\{ A_d p + (A_d^T - I_n) q + D_H^T A_d^T \zeta_2 \right\} + x_H^T(i, j) \left\{ -\zeta + D_H (\zeta_2 - \zeta_1) \right\} + x_s^T(i, j) \left\{ -D_H \zeta_1 \right\}\]

If condition (7) holds, one obtains

\[
(A^T - I_n) p + (A^T + D_H - D_L) q + \zeta + D_H^T ((A^T - I_n) \zeta_2 + \zeta_1 - D_H \zeta_2) < 0, \quad A_d^T p + (A_d^T - I_n) q + D_H^T A_d^T \zeta_2 < 0, \quad -\zeta + D_H (\zeta_2 - \zeta_1) < 0, \quad -D_H \zeta_1 < 0.
\]

It follows that \(\Delta V(i, j) < 0\), which means that

\[V^h(i + 1, j) + V^v(i, j + 1) < V^h(i, j) + V^v(i, j).\]

Summing up both sides of (16) from \(D\) to 0 with respect to \(i\) and from 0 to \(D\) with respect to \(j\), for any nonnegative integer
\[ D \geq \max(z_1, z_2) \], one gets
\[
V^h(1, D) + V^v(0, D + 1) + V^h(2, D - 1) \\
+ V^v(1, D) + \cdots + V^h(D + 1, 0) + V^v(D, 1) \\
= \sum_{i+j=D+1} V^h(i, j) + \sum_{i+j=D+1} V^v(i, j) \\
= \sum_{i+j=D} V(i, j) \\
\leq V^h(0, D) + V^v(0, D) + V^h(1, D - 1) \\
+ V^v(1, D - 1) + \cdots + V^h(D, 0) + V^v(D, 0) \\
= \sum_{i+j=D} V(i, j).
\]
Then from (9), we can conclude that
\[
\lim_{i+j \to \infty} x(i, j) = 0,
\]
which implies that the system (1a) and (1b) with \( w(i, j) = 0 \) is asymptotically stable.
This completes the proof. \( \square \)

When \( d_{ht} = d_{ht} = d_h \), and \( d_{vl} = d_{vl} = d_v \), the system (1a) and (1b) with \( w(i, j) = 0 \) is reduced to the following system:
\[
\begin{bmatrix}
\dot{x}^h(i+1, j) \\
\dot{x}^v(i, j+1)
\end{bmatrix} = A \begin{bmatrix}
\dot{x}^h(i, j) \\
\dot{x}^v(i, j)
\end{bmatrix} + A_d \begin{bmatrix}
\dot{x}^h(i-d_h, j) \\
\dot{x}^v(i-d_v, j)
\end{bmatrix},
\]
where \( d_h \) and \( d_v \) are constant delays along horizontal and vertical directions, respectively, and the boundary conditions are defined in (4). Then we can get the following result.

Corollary 5. For given positive constants \( d_h \) and \( d_v \), the positive 2D system (19) is asymptotically stable if there exist vectors \( \rho \) and \( q \in R^m \) such that
\[
\Phi = \text{diag} \{ \Phi_1, \Phi_2, \ldots, \Phi_n, \Phi'_1, \Phi'_2, \ldots, \Phi'_n \} < 0,
\]
where
\[
\Phi_k = (a_k^T - E_k) \rho + a_k^T q, \\
\Phi'_k = a_k^T \rho + (a_{jk}^T - E_k) q.
\]
with \( k \in \{ 1, 2, \ldots, n \} \), \( E_k = [0, \ldots, 0, 1, 0, \ldots, 0] \), \( p \), \( q \in R^m \), and \( a_k(a_{jk}) \) represents the \( k \)th column vector of matrix \( A(A_d) \).

Proof. Choose the following copositive Lyapunov-Krasovskii functional candidate for the system (19):
\[
V(i, j) = V^h(i, j) + V^v(i, j),
\]
where
\[
V^h(i, j) = \sum_{k=1}^2 V_k^h(i, j), \\
V^v(i, j) = x^T(i, j) \rho,
\]
with \( \rho^T, q^T \in R^m \), \( \rho^T, q^T \in R^m \). Then following the proof line of Theorem 4, the corollary can be obtained.

3.2. 1-Gain Analysis. The following theorem establishes sufficient condition of the asymptotic stability with 1-gain performance for the system (1a) and (1b).

Theorem 6. For given positive constants \( d_{hl}, d_{hl}, d_{vl}, d_{vl}, \) and \( y \), the positive 2D system (1a) and (1b) is asymptotically stable with the 1-gain index \( \gamma \) if there exist vectors \( p \in R^m \), \( q \in R^m \), \( \xi \in R^m \), and \( \zeta \in R^m \), such that
\[
\Phi = \text{diag} \{ \Phi_1, \Phi_2, \ldots, \Phi_n, \Phi'_1, \Phi'_2, \ldots, \Phi'_n \}, \\
\Phi'_1, \Phi'_2, \ldots, \Phi', n, T_1, T_2, \ldots, T_m < 0,
\]
where \( \Phi_k, \Phi'_k, \) and \( \Phi''_k \) are denoted as in Theorem 4, and
\[
\Phi_k = (a_k^T - E_k) \rho + (a_k^T + (d_{hl} - d_{vl}) E_k) q + E_k \xi \\
+ (d_{hl}^2 (a_k^T - E_k) - d_{hl} E_k) \xi_2, \\
1 \leq k \leq n_1, \\
+ d_{hl}^2 E_k \xi_1 + [E_k]_1, \\
\Phi'_k = a_k^T p + (a_{jk}^T - E_k) q, \\
n_1 + 1 \leq k \leq n_2, \\
+ d_{vl}^2 E_k \xi_1 + [E_k]_1, \\
T_1 = b_k^T \rho + b_k^T q + b_k^T D_{hl} \xi_2 + [E_k]_1 - \gamma, \\
1 \leq \varepsilon \leq m, \\
D_{hl} = \text{diag} \{ d_{hl} I_n, d_{hl} I_n \},
\]
with \( k \in \{ 1, 2, \ldots, n \} \), \( \varepsilon \in \{ 1, 2, \ldots, m \} \), \( E_k = [0, \ldots, 0, 1, 0, \ldots, 0] \), \( a_k, a_{jk} \), \( b_k \), and \( h_k \) represent the \( k \)th column vector of matrices \( A, A_d, B, \) and \( H \), respectively, and \( I_e \) represents the \( e \)th column vector of matrix \( L \).
Proof. It is an obvious fact that (24) implies the following inequality:

$$\Psi = \text{diag}\{\psi_1, \psi_2, \ldots, \psi_n, \psi'_1, \psi'_2, \ldots, \psi''_n, \psi''', \psi'''' \ldots\} < 0,$$

where

$$\Psi_k = \begin{cases}
(a_k^T - E_k)p + (a_k^T + (d_{ht} - d_{ht})E_k)q \\
+ \zeta_k + (d_{ht}^2(a_k^T - E_k) - d_{ht}E_k)\xi_k, & 1 \leq k \leq n_1, \\
(a_k^T - E_k)p + (a_k^T + d_{ht}E_k)q \\
+ \zeta_k + (d_{ht}^2(a_k^T - E_k) - d_{ht}E_k)\xi_k, & n_1 + 1 \leq k \leq n,
\end{cases}$$

$$\Psi'_k = \begin{cases}
(a_k^T - E_k)p + (a_k^T - E_k)q + (d_{ht}^2 - d_{ht})\xi_k, & 1 \leq k \leq n_1, \\
(a_k^T - E_k)p + (a_k^T - E_k)q + (d_{ht}^2 - d_{ht})\xi_k, & n_1 + 1 \leq k \leq n,
\end{cases}$$

$$\Psi''_k = \begin{cases}
-\zeta_k + d_{ht}E_k(\zeta_k - \xi_k), & 1 \leq k \leq n_1, \\
-\zeta_k + d_{ht}E_k(\zeta_k - \xi_k), & n_1 + 1 \leq k \leq n,
\end{cases}$$

$$\Psi'''_k = \begin{cases}
-d_{ht}E_k(\zeta_k), & 1 \leq k \leq n_1, \\
-d_{ht}E_k(\zeta_k), & n_1 + 1 \leq k \leq n.
\end{cases}$$

(27)

According to the definition of $l_1$ norm, one obtains

$$\|HX(i,j)\| = \sum_{k=1}^{p} h_{k,1} x^h_{1}(i,j) + \cdots + h_{k,n_1} x^h_{n_1}(i,j) + h_{k,n_1+1} x^h_{n_1+1}(i,j) + \cdots + h_{k,n} x^h_{n}(i,j),$$

$$= \left( \sum_{k=1}^{p} h_{k,1} \right) x^h_{1}(i,j) + \cdots + \left( \sum_{k=1}^{p} h_{k,n_1} \right) x^h_{n_1}(i,j) + \left( \sum_{k=1}^{p} h_{k,n_1+1} \right) x^h_{n_1+1}(i,j) + \cdots + \left( \sum_{k=1}^{p} h_{k,n} \right) x^h_{n}(i,j),$$

$$= x^T(i,j) \left[ \|h_1\|_1 \|h_2\|_1 \cdots \|h_p\|_1 \right]^T,$$

with $k \in \{1, 2, \ldots, n\}$, $E_k = \{0, \ldots, 1, 0, \ldots, 0\}$, and $a_k(a_{dk})$ represents the $k$th column vector of matrix $A(A_{dk})$.

By Theorem 4, we can obtain that the system (1a) and (1b) with $u(i,j) = 0$ is asymptotically stable. Now we are in a position to prove that the system (1a) and (1b) has a prescribed $l_1$-gain index $\gamma$ for any nonzero $u(i,j) \in l_1$.

To establish the $l_1$-gain performance, we choose the same copositive Lyapunov-Krasovskii functional candidate as in (9) for the system (1a) and (1b). Following the proof line of Theorem 4, we can get that

$$\Delta V(i,j) + \|z(i,j)\|_1 - \gamma \|w(i,j)\|_1,$$

(28)
where $h_k$ represents the $k$th column vector and $h_{i,j}$ represents the entry located at $(i, j)$ of matrix $H$. Then, similarly

$$
\|Lw (i, j)\|_1 = w^T (i, j) \left[ \|l_1\|_1 \; \|l_2\|_1 \; \cdots \; \|l_n\|_1 \right]^T, \\
\gamma \|w (i, j)\|_1 = w^T (i, j) [\gamma \; \gamma \; \cdots \; \gamma]^T,
$$

where $l_k$ represents the $k$th column vector of matrix $L$. Substituting (29)–(30) into (28) leads to

$$
\Delta V (i, j) + \|z(i, j)\|_1 - \gamma \|w (i, j)\|_1
= x^T (i, j) \left\{ (A^T - I_n) p + (A^T + D_H - D_L) q + \zeta \\
+ D_H^2 \left( (A^T - I_n) \varsigma_2 + \varsigma_1 \right) - D_{L}\varsigma_2 \\
+ \left[ \|l_1\|_1 \; \|l_2\|_1 \; \cdots \; \|l_n\|_1 \right]^T \right\}
+ x_{d}(i, j) \left[ A_H^T p + (A_H^T - I_n) q + D_H^T A_H^T \varsigma_2 \right]
+ x_{d}^T (i, j) \left[ -\zeta + D_{L}\varsigma_2 - \varsigma_1 \right]
+ x_{d}^T (i, j) \left[ -D_{L}\varsigma_1 \right]
+ w^T (i, j) \left[ B^T (p + q + D_H^T \varsigma_2) \right]
+ \left[ \|l_1\|_1 \; \|l_2\|_1 \; \cdots \; \|l_n\|_1 \right]^T
- [\gamma \; \gamma \; \cdots \; \gamma]^T \right\}.
$$

If condition (24) holds, we have

$$
V^h (i + 1, j) - V^h (i, j) + V^v (i, j + 1) - V^v (i, j)
+ \|z (i, j)\|_1 - \gamma \|w (i, j)\|_1 < 0.
$$

(32)

We know that

$$
\Delta V (i, j) = V^h (i + 1, j) - V^h (i, j) + V^v (i, j + 1) - V^v (i, j).
$$

(33)

For any positive scalars $k_h$ and $k_v \in \mathbb{Z}_+$, it can be verified that

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V (i, j)
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( V^h (i + 1, j) - V^h (i, j) \right)
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( V^v (i, j + 1) - V^v (i, j) \right)
= \sum_{j=0}^n \left( V^h (k_h + 1, j) - V^h (0, j) \right)
+ \sum_{i=0}^{\infty} \sum_{j=0}^{k_v} \left( V^v (i, k_v + 1) - V^v (i, 0) \right).
$$

When $k_h$ and $k_v = \infty$, we have

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \|z (i, j)\|_1 - \gamma \|w (i, j)\|_1 \right) < \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V (i, j).
$$

(35)

The existence of solution for LMI (24) implies that the positive 2D system (1a) and (1b) is asymptotically stable. Together with the zero boundary conditions, one can get

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V (i, j) = 0.
$$

(36)

Applying (36) to (35), one has

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|z (i, j)\|_1 < \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|w (i, j)\|_1.
$$

(37)

By Definition 2, the positive 2D system (1a) and (1b) is asymptotically stable and has the $l_1$ -gain index $\gamma$.

This completes the proof. $\square$

Remark 7. In Theorem 6, the disturbance attenuation performance of positive 2D linear systems is analyzed, and sufficient conditions for the existence of $l_1$ -gain performance for positive 2D system to (1a) and (1b) are proposed in terms of LMIs which are computationally tractable. This is also the major contribution of our paper.

4. Numerical Example

Consider the positive 2D system with delays in the Roesser model (1a) and (1b), where

$$
A = \begin{bmatrix}
0.10 & 0.20 & : & 0.2 \\
0.00 & 0.30 & : & 0.10 \\
\cdots & \cdots & \cdots & \cdots \\
0.00 & 0.10 & : & 0.40
\end{bmatrix},
A_d = \begin{bmatrix}
0.10 & 0.01 & : & 0.05 \\
0.10 & 0.02 & : & 0.05 \\
\cdots & \cdots & \cdots & \cdots \\
0.03 & 0.12 & : & 0.03
\end{bmatrix},
B = \begin{bmatrix}
0.2 \\
0.1 \\
0.1
\end{bmatrix},
H = \begin{bmatrix}
0.1 & 0.2 \\
0.1 & 0.1
\end{bmatrix},
L = 0.1,
d_h (i) = 4 + 2 \sin \left( \frac{\pi i}{2} \right),
d_v (j) = 5 + 2 \sin \left( \frac{\pi j}{2} \right),
\omega (i, j) = e^{-(i+0.5j)},
$$

(38)
where state dimensions are $n_h = 2$ and $n_v = 1$. The boundary conditions are given by
\[
\begin{align*}
x^h(i, j) &= \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, & \forall 0 \leq j \leq 52, -d_{hL} \leq i \leq 0, \\
x^v(i, j) &= 0.1, & \forall 0 \leq i \leq 52, -d_{vL} \leq j \leq 0.
\end{align*}
\] (39)

In this example, we can get $d_{hL} = 2$, $d_{hH} = 6$, $d_{vL} = 3$, and $d_{vH} = 7$. Given $\gamma = 4.5$, then by using the LMI Control Toolbox [37] to solve the inequalities in Theorem 6, we can get the following solutions:
\[
\begin{align*}
p &= \begin{bmatrix} 1.7310 & 1.6842 & 1.7364 \\ 3.2077 & 2.8768 & 3.3026 \end{bmatrix}^T, \\
q &= \begin{bmatrix} 1.0334 & 1.1075 & 0.9957 \end{bmatrix}^T, \\
\zeta &= \begin{bmatrix} 0.3675 & 0.4042 & 0.3552 \\ 0.0385 & 0.0489 & 0.0330 \end{bmatrix}^T.
\end{align*}
\] (40)

Figures 1, 2, and 3 show the state responses of the system; it can be seen that the corresponding positive 2D system is asymptotically stable. Furthermore, by computing, under zero boundary conditions, we have $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|z(i, j)\|_1 = 4.0206 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|w(i, j)\|_1 = 1.0977$. It is obvious that the prescribed $l_1$-gain performance level $\gamma = 4.5$ is satisfied.

5. Conclusions

This paper has addressed the delay-dependent stability analysis with $l_1$-gain performance for positive 2D systems with state delays in the Roesser model. A sufficient condition for the existence of the delay-dependent asymptotic stability of positive 2D linear systems with time delays has been established. Copositive-type Lyapunov function method has been used to get a computationally tractable LMI-based sufficient criterion which ensures that the system is asymptotically stable and has a prescribed $l_1$-gain performance. A numerical example has been given to illustrate the efficiency of the results. Furthermore, our future work will be devoted to the $l_1$-gain control problem for positive 2D systems with delays.

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