Stability Analysis with H_∞ performance of production networks with autonomous work systems and time-varying delays

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***Abstract*— This paper considers the problem of stability analysis with an H_∞ performance for a class of production networks of autonomous work systems with delays in the capacity changes. The system under consideration shares information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of local work in progress. An appropriate Lyapunov function is utilized to establish some delay-range-dependent conditions in terms of linear matrix inequalities for the stability analysis of the network.**

I. INTRODUCTION

Production networks are emerging as a new type of cooperation between and within companies, requiring new techniques and methods for their operation and management [1]. However, the dynamic and structural complexity of these emerging networks inhibits collection of the information necessary for centralized planning and control, and decentralized coordination must be provided by logistic processes with autonomous capabilities [2]. The behaviour of a production network is affected by external and internal order flows, planning, internal disturbances, and the control laws used locally in the work systems to adjust resources for processing orders [3]. In prior work, sharing of capacity information between work systems has been modelled in [4] along with the benefits of alternative control laws and reducing delay in capacity changes [5]-[6]. Several authors have described both linear and nonlinear dynamical models for control of variables such as inventory levels and work in progress (WIP); see for instance [7]. Understanding the dynamic nature of production systems requires new approaches for the design of Production Planning and Control (PPC) based on company's logistics. The controllers implicitly interact to adjust capacity to eliminate backlog as the system maintains its planned WIP level; see [8]-[10]. A discrete closed-loop PPC model was developed and analyzed by Duffie and Falu [11] in which two discrete controllers, one for backlog and one for WIP, with different periods between adjustments of work input and capacity, respectively, were selected and evaluated using transfer function analysis and time-response simulation.

In this paper, we contribute to the problem of stability analysis with an H_∞ performance for a class of production networks of autonomous work systems with delays in the capacity changes. The system under consideration shares information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of WIP. The contribution of this paper is three-fold: first, this paper extends previous works on the stability analysis problem of a production network of autonomous work systems with a time-varying delay within the capacity changes and derives some new theoretical results; second, this paper shows how the stability analysis problem can be reduced to a convex problem with additional degrees of freedom; third, by using a Lyapunov function, we establish new required sufficient conditions in terms of delay-range-dependent linear matrix inequalities (LMIs) under which the network satisfies both asymptotic stability and a prescribed level gain. Finally, numerical results are given to illustrate the usefulness of the proposed methodology.

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II. DYNAMIC MODEL

Assume that there are N work systems in a production network and that vector $i(nT)$ is the rate at which orders are input to the N work systems from sources external to the production network, which is constant over time $nT \le t < (n+1)T$ where $n = 1, 2, \cdots$ and T is a time period between capacity adjustments. The total orders that have been input to the work systems up to time $(k+1)T$ then can be represented as the vector [5]

$$w_i((n+1)T) = w_i(nT) + T(i(nT) + P(nT)^T c_a(nT)) \qquad (1)$$

where vector $c_a(nT)$ is the rate at which orders are output from the N work systems during time $nT \le t < (n+1)T$ (the actual capacity of each work system) and $P(nT)$ is a matrix in which each element $p_{ij}(nT)$ represents the actual fraction of the orders flowing out of work system i that flow into work system j during time $nT \le t < (n+1)T$. $P(nT)$ is assumed to be constant duing this period. The total number of orders that have been output by the work systems up to time $nT \le t < (n+1)T$ can be represented by the vector

$$w_o((n+1)T) = w_o(nT) + Tc_a(nT) \qquad (2)$$

while the rate at which orders are output from the network during time $nT \le t < (n+1)T$ is

$$o(nT) = P_o(nT)c_a(nT) \qquad (3)$$

where $P_o(nT)$ is a matrix in which non-zero element $p_o(nT)_{ii}$ represents the fraction of orders flowing out of work system i that flow out of the network during time $nT \le t < (n+1)T$. $p_o(nT)_{ij}$ is assumed to be constant duing this period, and

$$p_{oii}(nT) + \sum_{j=1}^{N} p_{oij}(nT) = 1 \qquad (4)$$

The WIP in the work systems is

$$wip_a(nT) = w_i(nT) - w_o(nT) + w_d(nT) \quad (5)$$

where $w_d(nT)$ represents local work disturbance, such as rush order, that affect the work system. Consider a network in which information is shared between work systems and in which capacity plans are supplied in advance by a source external to the network. Furthermore, the actual capacity of each work system is as follows:

$$c_a(nT) = Q(nT)c_m(nT) - c_d(nT) \quad (6)$$

where $c_d(nT)$ represents local capacity disturbances such as equipment failures, $Q(nT) = (I - P(nT))^{-1}$ indicates the information coupling between the work systems and $Q(nT)$, a matrix in which each element $q(nT)_{ij}$ represents the expected fraction of the orders flowing out of work system i that flow into work system j. In addition, $c_m(nT)$ represents local capacity adjustments to maintain the WIP in each work system in the vicinity of the planned levels $wip_p(nT)$ using straightforward proportionality k_c and is described in the form of

$$c_m(nT) = i'(nT)\Delta C((n - d(n))T) \quad (7)$$

where $i'(nT)$ is expected input rates from sources external to the network and

$$\Delta C(nT) = k_c(wip_a(nT) - wip_p(nT)) \quad (8)$$

It is assumed that a time-varying delay exists in the capacity changes $c_m(nT)$ for logistic reasons such as operator work rules and satisfies

$$0 < d_1 \le d(n) \le d_2 \quad (9)$$

and the planned capacity and WIP are also assumed to be known and delay free in advance.

Eqs. (1)-(8) can be combined to obtain a discrete-time model for the system:

$$X_i((n+1)T) = X_i(nT) + BX_i((n - d(n))T) + CW(nT) \quad (10)$$

$$o(nT) = DX_i((n - d(n))T) + EW(nT) \quad (11)$$

where $X_i(nT) = \begin{bmatrix} w_i(nT)^T & w_o(nT)^T \end{bmatrix}^T$, and

$$B = \begin{bmatrix} Tk_cP^TQ & -Tk_cP^TQ \\ TQ & -TQ \end{bmatrix}$$

$$C = \begin{bmatrix} TI & TP^TQ & -TP^T & Tk_cP^TQ & -Tk_cP^TQ \\ 0 & TQ & -TI & Tk_cQ & -Tk_cQ \end{bmatrix}$$

$D = \begin{bmatrix} k_cP_oQ & -k_cP_oQ \end{bmatrix}$, $E = \begin{bmatrix} 0 & k_cP_oQ & -P_o & k_cP_oQ & -k_cP_oQ \end{bmatrix}$.

The stability analysis problem with an H_∞ performance to be addressed in this paper can be formulated such that

1) The system (10)-(11) is asymptotically stable when $W(nT) = 0$.
2) Under the zero-initial condition and for any nonzero $W(nT)$ with a prescribed scalar $\gamma > 0$, the output $o(nT)$ satisfies the following H_∞ performance measure

$$\|o(nT)\|_2 \le \gamma \|W(nT)\|_2 \quad (12)$$

III. Stability analysis with H_∞ performance

In this section, by assuming that the control gain k_c is known, some delay-range-dependent conditions for the stability of the network respect to the delay parameter are proposed using Lyapunov method.

Theorem 3.1: For given positive scalars d_1, d_2, γ , the system (10)-(11) is asymptotically stable and satisfies the H_∞ performance bound γ by the control gain k_c , if there exist a matrix Y and some positive-definite matrices S_1, S_2, S_3, Z such that the following LMIs are feasible,

$$\Gamma := \begin{bmatrix} \Gamma_{11} & -Y + S_1B & S_1C \\ * & \Gamma_{22} & \Gamma_{23} \\ * & * & \Gamma_{33} \end{bmatrix} < 0 \quad (13)$$

$$\begin{bmatrix} S_1 & Y \\ * & Z \end{bmatrix} \ge 0 \quad (14)$$

with $d_{12} := d_2 - d_1$ and $\Gamma_{11} = Y + Y^T + S_3(1 + d_{12}) + d_2S_1$, $\Gamma_{22} = -S_3 + B^TS_1B + D^TD$, $\Gamma_{23} = B^T(d_2Z + S_1)C + D^TE$, $\Gamma_{33} = -\gamma^2I + C^T(S_1 + d_2Z)C + E^TE$. The symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix.

Proof: Consider the Lyapunov function candidate in the following form

$$V(nT) = \sum_{i=1}^{4} V_i(nT) \quad (15)$$

where

$$V_1(nT) = X_i(nT)^TS_1X_i(nT) \quad (16)$$

$$V_2(nT) = \sum_{s=-d_2}^{-1} \sum_{j=k+s+1}^{k} [X_i(jT) - X_i((j-1)T)]^T \times S_2[X_i(jT) - X_i((j-1)T)] \quad (17)$$

$$V_3(nT) = \sum_{j=n-d(n)}^{n-1} X_i(jT)^TS_3X_i(jT) \quad (18)$$

$$V_4(nT) = \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=n+j-1}^{n-1} X_i(lT)^TS_3X_i(lT) \quad (19)$$

After some manipulations on $\Delta V_i(nT) = V_i((n+1)T) - V_i(nT)$, the following result can be obtained

$$\Delta V(nT) = \sum_{i=1}^{4} \Delta V_i(nT) \le \chi(nT)^T\Gamma_1\chi(nT) \quad (20)$$

where $\chi(nT) := \begin{bmatrix} \hat{\chi}(nT)^T & W(nT)^T \end{bmatrix}$ and

$$\Gamma_1 := \begin{bmatrix} \hat{\Gamma}_1 & \begin{matrix} S_1C \\ B^T(d_2Z + S_1)C \end{matrix} \\ * & C^T(S_1B^T + d_2Z)C \end{bmatrix}$$

with

$$\hat{\chi}(nT) := \begin{bmatrix} X_i(nT)^T & X_i((n - d(n))T)^T \end{bmatrix}$$

$$\hat{\Gamma}_1 := \begin{bmatrix} \Gamma_{11} & -Y + S_1B \\ * & -S_3 + B^TS_1B \end{bmatrix}$$

Furthermore, in the case of $W(nT) = 0$, it follows from (20) that

$$\Delta V(nT) \leq \hat{\chi}(nT)^T \hat{\Gamma}_1 \hat{\chi}(nT) \leq -\lambda_{min}(-\hat{\Gamma}_1) |\hat{\chi}(nT)|^2 \quad (21)$$

On the other hand, considering the Lyapunov function (15), one gets

$$\lambda_{min}(S_1) |X_i(nT)|^2 \leq V(nT)$$

$$\leq \alpha |X_i(nT)|^2 + \alpha(1 + d_{12} + 2d_2) \sum_{j=n-d_2}^{n} |X_i(jT)|^2 \quad (22)$$

where $\alpha = max\{\lambda_{max}(s_1), \lambda_{max}(s_2), \lambda_{max}(s_3))\}$. Define

$$J_M = \sum_{n=0}^{M} [o(nT)^T o(nT) - \gamma^2 W(nT)^T W(nT)] \quad (23)$$

where M is a positive integer scalar. Now, noting the zero initial condition and (20), one has

$$J_M = \sum_{n=0}^{M} [o(nT)^T o(nT) - \gamma^2 W(nT)^T W(nT) + \Delta V(nT)]$$

$$-V((M+1)T) \leq \sum_{n=0}^{\infty} \chi(nT)^T \Gamma \chi(nT)$$

with Γ defined in (13). Now, it follows from the inequality above that $\Gamma < 0$, which together with (22) ensure that (12) holds under the zero initial condition. $\diamond$

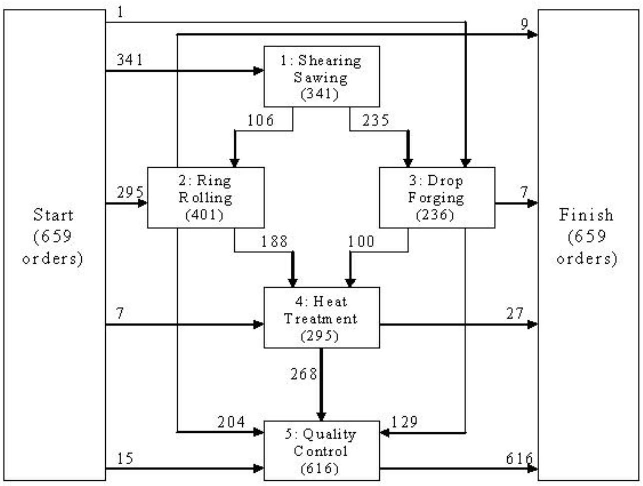

Fig. 1. Omni-directional order-flow structure example.

IV. Numerical Results

Consider the case of a supplier of components to the automotive industry and for which production data documents orders flowing between five work systems over a 162-day period. These work systems and the order-flow structure over this period is illustrated in Fig. 1. In this network, all order flows are unidirectional. Then, the internal flow of orders is approximated using the following matrix [3],

$$P = \begin{bmatrix} 0 & 106/341 & 235/341 & 0 & 0 \\ 0 & 0 & 0 & 188/401 & 204/401 \\ 0 & 0 & 0 & 100/236 & 129/236 \\ 0 & 0 & 0 & 0 & 268/295 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the sampling time $T = 1$ scd. According to Theorem 3.1, for the asymptotic stability of the network under consideration, H_∞ performance levels, γ, under $d_1 = 1$, the controller gain $k_c = 0.1\, scd^{-1}$ and different values of the upper bounds of the delay, d_2, are shown in Table I.

TABLE I

Optimal H_∞ performance levels γ w.r.t. d_2.

	$d_2 = 2$	$d_2 = 3$	$d_2 = 4$	$d_2 = 5$
Theorem 3.1	0.385	0.475	0.730	0.785

V. Conclusion

The problem of stability analysis with an H_∞ performance for a class of production networks of autonomous work systems with delays in the capacity changes was investigated in this paper. The system under consideration shares information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of local work in progress. In terms of linear matrix inequalities some delay-range-dependent stability conditions were derived for the network.

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